MA 509 - REAL ANALYSIS - LECTURE 32

Thm. 4.11 If f is a cont. mapping of a cpt. m.s. X into $\mathbb{R}^k$, then $f(X)$ is closed and bounded. Thus, f is bounded.

Proof:- By the previous theorem, $f(X)$ is a cpt. subset of $\mathbb{R}^k$. Hence by Heine-Borel theorem, $f(X)$ is closed and bounded.
Hence $f(X)$ is bounded.

$\blacksquare$

Thm. 4.12 Suppose f is a cont. real-valued function on a cpt. m.s. X, and

$$M = \sup_{p\in X}(f(p)), \quad m = \inf_{p\in X}(f(p)),$$

then $\exists$ pts. $p, q \in X \ni f(p) = M$, $f(q) = m$.

Proof: By the prev. thm., $f(X)$ is closed and bounded subset of $\mathbb{R}$. Hence by Thm. 2.16 (Lec. 13),

$$M = \sup(f(X)) \in f(X)$$
$$m = \inf(f(X)) \in f(X).$$

Thm. 4.13 Suppose f is a cont. 1-1 mapping of a cpt. m.s. X onto a m.s. Y. Then the inverse mapping f^{-1} defined on Y is a continuous function on Y.

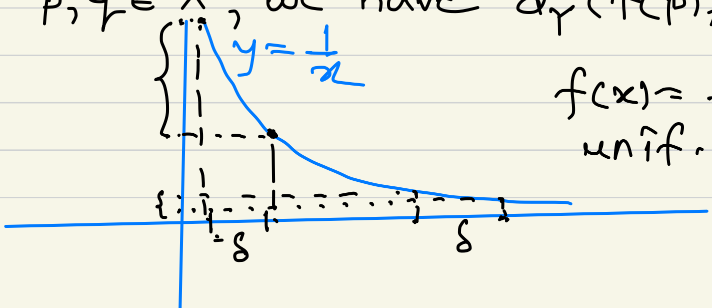
Proof:— Claim: For every open V of X, $f(V)$ is open in Y.

To that end, fix an open set V in X. Then V^c is a closed subset of cpt. m.s. X, hence V^c is cpt.
But then by Thm. 4.11, $f(V^c)$ is cpt. subset of Y, hence closed.

But f is 1–1 & onto. Hence
$$f(V^c) = f(V)^c$$
Let $y \in f(V^c) \Rightarrow \exists!\, x \in V^c \ni f(x) = y$.
Also $x \notin V \Rightarrow f(x) \notin f(V)$ so that $f(x) \in f(V^c)$.
So $f(V^c) \subseteq f(V)^c$. Similarly show $f(V^c) \subseteq f(V^c)$.
$\Rightarrow f(V)$ is open in Y.
$\Rightarrow f^{-1}: Y \to X$ is cont. on Y. ∎

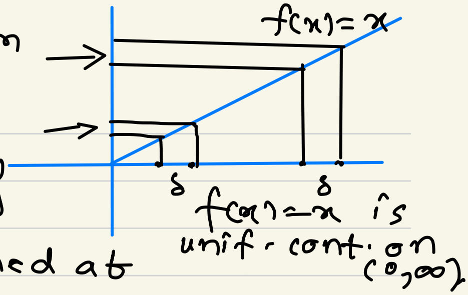
UNIFORM CONTINUITY

Let f be a mapping from a metric space into a metric space Y. Then f is said to be uniformly continuous on X if for every $\varepsilon > 0$, $\exists\, \delta > 0 \ni$ whenever $d_X(p,q) < \delta$ for any $p, q \in X$, we have $d_Y(f(p), f(q)) < \varepsilon$.



$f(x) = \frac{1}{x}$ is not unif. cont. on $(0, \infty)$.

Differences between uniform continuity & continuity



① Unif. cont. is a property of a function on a set, whereas cont. can be defined at a single point.

② In unif. cont., δ is a function of only ε, whereas in cont. δ is a function of both ε & the point where it is continuous.

* Uniformly continuous function is continuous

Thm. 4.14 Let f be a cont. mapping of a compact m.s. X into a m.s. Y. Then f is uniformly continuous on X.

Proof: Given $\varepsilon > 0$, f cont. implies that associated to a pt. $p \in X$, $\exists\ \phi(p) > 0 \ni$
$q \in X$, $d_X(p,q) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$ — (*)

Let $J(p) = \{ q \in X : d_X(p,q) < \frac{1}{2}\phi(p) \}$.

Note that $p \in J(p)$. Hence $\{J(p)\}$, where $p \in X$ forms an open cover of X.

Since X is cpt., $\exists\ p_1, p_2, \ldots, p_n$ in $X \ni$

$$X \subset J(p_1) \cup J(p_2) \cup \cdots \cup J(p_n). \quad \text{— (**)}$$

Now let $\delta = \frac{1}{2} \min\{ \phi(p_1), \phi(p_2), \ldots, \phi(p_n) \}$.

Then $\delta > 0$.

Now we show unif. cont. of f on X.
Let $q, p \in X \ni d_X(p,q) < \delta$. By (**),
$\exists\, m$, $1 \le m \le n \ni p \in J(p_m)$ so that

$$d_X(p, p_m) < \frac{1}{2}\phi(p_m).$$

Also, $d_X(q, p_m) \le d_X(p,q) + d_X(p, p_m)$

$$< \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m).$$

Hence by (*),

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow f$ is u.c. on X. $\blacksquare$

Thm. 4.15 Let E be a non-compact set in $\mathbb{R}$.
Then:

(a) $\exists$ a cont. fn. on E which is not bounded.

(b) $\exists$ a cont. and bdd. fn. on E which has no maximum.

In addition to the above hypotheses, let E be bounded. Then,

(c) $\exists$ a cont. fn. on E which is not u.c.