11/11/2020

MA 509 - REAL ANALYSIS - LECTURE 35

Infinite limits and limits at infinity

- If $x \in \mathbb{R}$, a nbhd of x is a segment $(x-\delta, x+\delta)$ for $\delta > 0$.

- **Defn:** For any $c \in \mathbb{R}$, the set of real numbers x s.t. $x > c$ is a called a neighborhood of $+\infty$, and written as $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a nbhd of $-\infty$.

Defn. Let f be a real fn. defined on E. Then
$f(t) \to A$ as $t \to x$ where A & x are in $\mathbb{R} \cup \{\pm\infty\}$, if for every nbhd U of A, there is a nbhd V of x s.t. $V \cap E$ is not empty, and s.t. $f(t) \in U$ $\forall$ $t \in V \cap E$, $t \neq x$.

Chapter 5 – Differentiation

- The derivative of a real function:

Let f be real-valued on $[a,b]$. For any $x \in [a,b]$, let $\phi(t) = \dfrac{f(t)-f(x)}{t-x}$ $(a<t<b,\ t \neq x)$

(difference quotient)

and define $f'(x) = \lim_{t \to x} \phi(t)$, provided this limit exists.

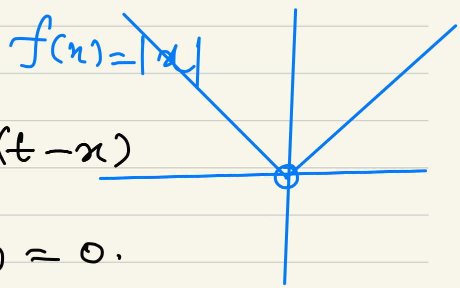
Also, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$.

* f' is called the derivative of f.

- At the end points a or b, the derivative, if it exists, is a right-hand or left-hand derivative, respectively.

DIFFERENTIABILITY IMPLIES CONTINUITY

Thm. 5.1 Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.



Proof: As $t \to x$,

$$f(t)-f(x) = \frac{f(t)-f(x)}{t-x} \cdot (t-x) \longrightarrow f'(x) \cdot 0 = 0.$$

$\Rightarrow \lim_{t \to x} f(t) = f(x)$. Hence f is cont. at x.

$$\lim_{t \to 0^+} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^+} \frac{|t|}{t} = \lim_{t \to 0^+} \frac{t}{t} = 1,$$

$$\lim_{t \to 0^-} \frac{f(t)-f(0)}{t-0} = \lim_{t \to 0^-} \frac{|t|}{t} = \lim_{t \to 0^-} \frac{-t}{t} = -1$$

- Rules of differentiation
- Derivatives of elementary functions

Thm. 5.2 Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ $(a \leq t \leq b)$, then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

Proof: Let $y = f(x)$. Using the defn. of derivative, we have

$$f(t) - f(x) = (t-x)(f'(x) + u(t)) \quad \text{— (a)}$$
$$g(s) - g(y) = (s-y)(g'(y) + v(s)), \quad \text{— (b)}$$

where $t \in [a,b]$, $s \in I$, and $u(t) \to 0$, $v(s) \to 0$ as $s \to y$.

Let $s = f(t)$. From (a) & (b),

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= (f(t) - f(x))(g'(y) + v(s)) \\ &= (t-x)(f'(x) + u(t))(g'(y) + v(s)). \end{aligned}$$

Hence for $t \neq x$,

$$\frac{h(t) - h(x)}{t-x} = (g'(y) + v(s))(f'(x) + u(t)).$$

Now let $t \to x$. Then $s \to y$ (by the continuity of f).

$\Rightarrow h'(x) = g'(f(x))f'(x)$. ∎

Examples

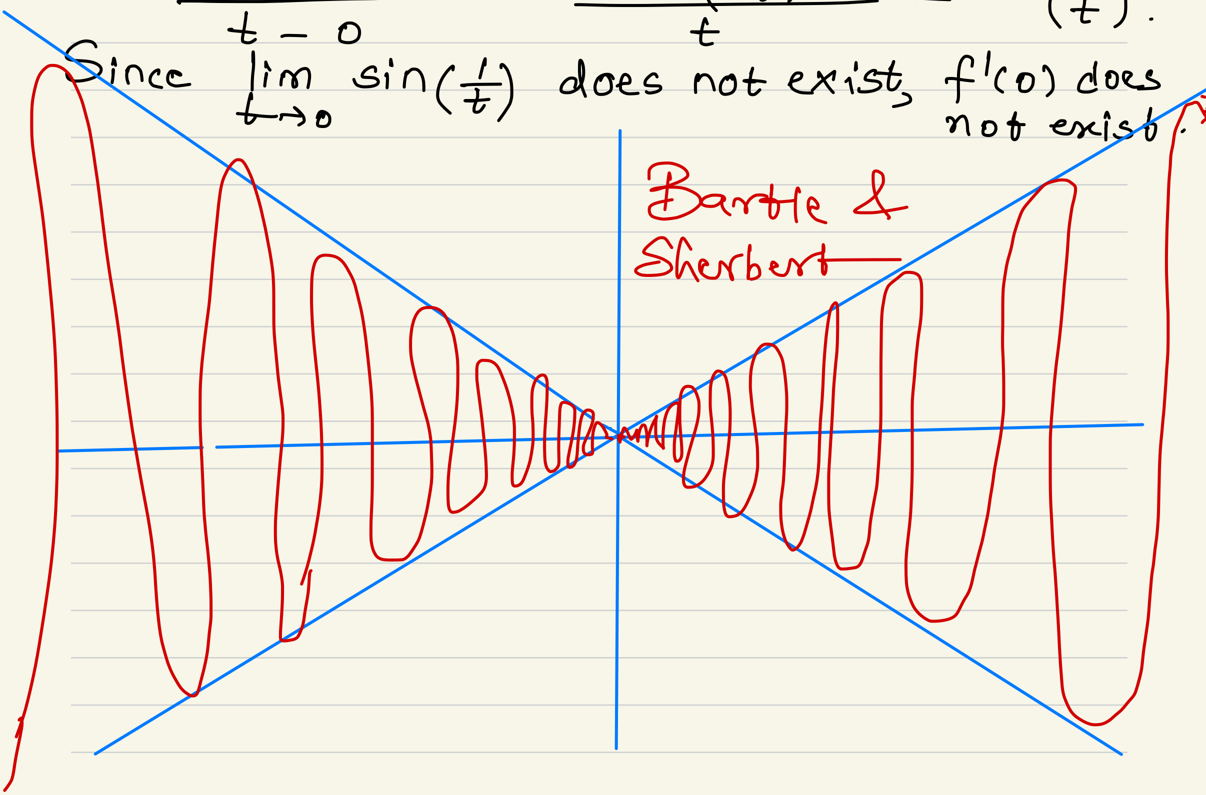
① $f(x) = \begin{cases} x\sin(1/x), & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

Whenever $x \neq 0$, $f'(x) = x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \sin(1/x)$ exists.

At $x = 0$, we use the actual defn. of derivative. Note that for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \frac{t\sin(1/t) - 0}{t} = \sin\left(\frac{1}{t}\right).$$

Since $\lim_{t \to 0} \sin\left(\frac{1}{t}\right)$ does not exist, $f'(0)$ does not exist.



② $f(x) = \begin{cases} x^2 \sin(1/x), & (x \neq 0) \\ 0, & (x = 0) \end{cases}$

$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \quad (x \neq 0).$

At $x = 0$: $\dfrac{f(t) - f(0)}{t - 0} = \dfrac{t^2 \sin\left(\frac{1}{t}\right) - 0}{t} = t \sin\left(\frac{1}{t}\right).$

$$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} t \sin\left(\frac{1}{t}\right) = 0.$$

$\left(\because \left| t \sin\left(\frac{1}{t}\right) \right| \leq |t| \right)$

$\Rightarrow$ f is differentriable for every $x \in \mathbb{R}$.

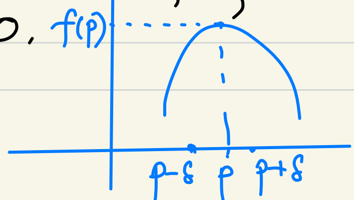
But f' is not continuous at $x = 0$.

<u>MEAN VALUE THEOREM</u>

<u>Defn.</u> Let f be a real function defined on a metric space X. We say f has a local maximum at a point $p \in X$ if $\exists\, \delta > 0 \ni f(q) \leq f(p)\ \forall q \in X$ with $d(p, q) < \delta$.

Similarly, we define local minima.

<u>Thm. 5.3</u> Let f be defined in $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.



Proof: Choose $\delta \ni \quad a < x-\delta < x < x+\delta < b$.

If $x-\delta < t < x$, then
$$\frac{f(t)-f(x)}{t-x} \geq 0$$
Now let $t \to x$, so that $f'(x) \geq 0$. ——— (1)

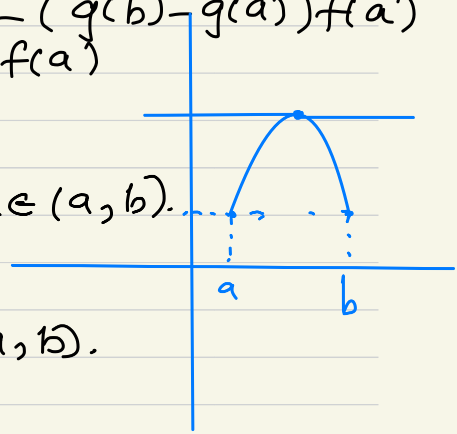
If $x < t < x+\delta$, then $\dfrac{f(t)-f(x)}{t-x} \leq 0$ so that $f'(x) \leq 0$ ——— (2)

Hence from (1) & (2), $f'(x) = 0$.

Thm. 5.4 If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then $\exists\, x \in (a,b) \ni$
$$(f(b)-f(a))\,g'(x) = (g(b)-g(a))\,f'(x).$$

Proof: Let $h(t) := (f(b)-f(a))g(t) - (g(b)-g(a))f(t)$, for $a \leq t \leq b$.

Then h is clearly continuous on $[a,b]$, & differentiable on (a,b), and
$$\begin{aligned} h(a) &= (f(b)-f(a))g(a) - (g(b)-g(a))f(a) \\ &= f(b)g(a) - g(b)f(a) \\ &= h(b). \end{aligned}$$



Claim: $h'(x) = 0$ for some $x \in (a,b)$.

Case 1: h is constant.

Then $h'(x) = 0 \;\forall\, x \in (a,b)$.

Case 2

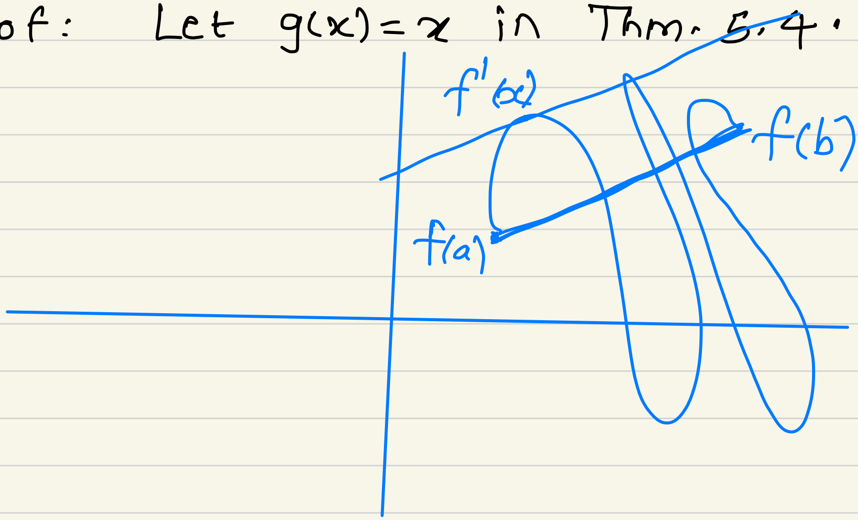
If $h(t) > h(a)$ for some $t \in (a,b)$, let x be a point in $[a,b]$ at which h attains its maximum. Then $h(a) = h(b)$ implies $x \in (a,b)$ & then $h'(x) = 0$.

Case 3 If $h(t) < h(a)$ for some $t \in (a,b)$, choose $x \in [a,b]$ at which h attains its maximum.

Cor. 5.5 If f is a continuous real function on $[a,b]$ which is differentiable in (a,b), $\exists\, x \in (a,b) \ni$

$$f(b) - f(a) = (b-a) f'(x).$$

Proof: Let $g(x) = x$ in Thm. 5.4.



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