18/11/2020

MA 509 - REAL ANALYSIS - LECTURE 38

A version of the mean-value theorem for vector-valued functions

Thm. 5.11 Suppose $\bar{f}$ is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and $\bar{f}$ is differentiable in (a,b). Then $\exists\, x \in (a,b) \ni$

$$|\bar{f}(b) - \bar{f}(a)| \leq (b-a)|\bar{f}'(x)|.$$

Proof: Let $z = \bar{f}(b) - \bar{f}(a)$.

Define $\varphi(t) = z \cdot \bar{f}(t) \qquad (a \leq t \leq b)$.

Then φ is a real-valued cont. fn. on $[a,b]$ diff. in (a,b). Hence by the usual mean-value theorem,

$$\varphi(b) - \varphi(a) = (b-a)\varphi'(x) = (b-a)\, z \cdot \bar{f}'(x)$$

for some $x \in (a,b)$.

But $\varphi(b) - \varphi(a) = z \cdot \bar{f}(b) - z \cdot \bar{f}(a)$
$= z \cdot z = |z|^2$.

By Cauchy-Schwarz inequality,

$$|z|^2 = (b-a)|z \cdot f'(x)| \leq (b-a)|z|\,|\bar{f}'(x)|.$$

Hence $|z| \leq (b-a)|\bar{f}'(x)|$.

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Chapter 6 – The Riemann-Stieltjes integral

RIEMANN INTEGRAL: Defn. & existence

Consider the interval $[a,b]$. By a partition P of $[a,b]$, we mean a finite set of points $x_0, x_1, \dots, x_n$, where $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$.

Let $\Delta x_i = x_i - x_{i-1}$ $(i = 1, 2, \dots, n)$.
Suppose f is a bounded real function of $[a,b]$.
Set
$$M_i := \sup_{x_{i-1} \le x \le x_i} f(x)$$
$$m_i := \inf_{x_{i-1} \le x \le x_i} f(x) .$$
$$U(P,f) := \sum_{i=1}^{n} M_i \, \Delta x_i$$
$$L(P,f) := \sum_{i=1}^{n} m_i \, \Delta x_i ,$$
and
$$\overline{\int_a^b} f \, dx := \inf_{P \in [a,b]} U(P,f),$$
Upper Riemann integral of f over $[a,b]$

$$\underline{\int_a^b} f \, dx := \sup_{P \in [a,b]} L(P,f) .$$
lower Riemann integral of f over $[a,b]$

If $\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx$, we say f is Riemann integrable on $[a,b]$, and we write $f \in \mathcal{R}$, where $\mathcal{R}$ denotes the set of Riemann integrable functions.

The Riemann integral of f is denoted by
$\int_a^b f\,dx$ or by $\int_a^b f(x)\,dx$.

Since f is given to be bounded, $\exists\ m, M \in \mathbb{R} \ni$
$$m \le f(x) \le M \qquad (a \le x \le b).$$

This implies that <u>for every partition P</u>,

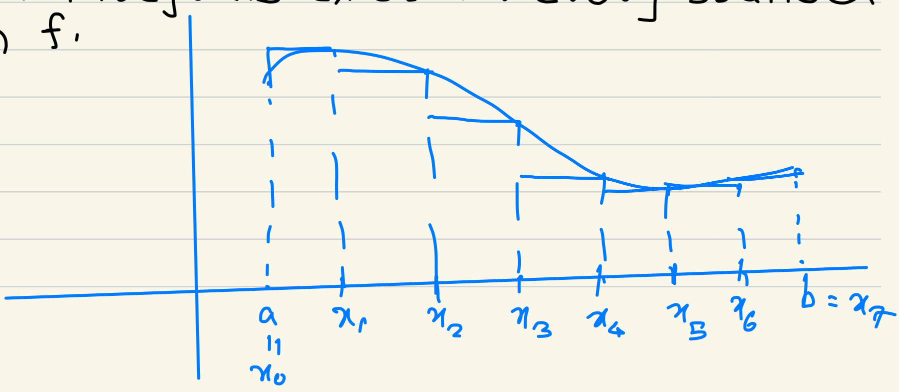
$$\underline{m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)}.$$

Proof: $M_i = \sup\limits_{[x_{i-1},x_i]} f(x) \le M \qquad (1 \le i \le n)$

$$\Rightarrow \sum_{i=1}^{n} M_i \Delta x_i \le \sum_{i=1}^{n} M \Delta x_i = M \sum_{i=1}^{n} \Delta x_i = M(\Delta x_n - \Delta x_0)$$

i.e., $U(P,f) \le M(b-a)$.
Similarly, $m(b-a) \le L(P,f)$
But $L(P,f) \le U(P,f)$ Hence the claim.
$(\because m_i \le M_i \ \forall i)$

Thus, the numbers $L(P,f)$ and $U(P,f)$ form a bounded set, and that the upper & lower Riemann integrals exist for every bounded function f.



STIELTJES INTEGRAL

Let α be a monotonically increasing function on $[a,b]$. Note that since $\alpha(a)$ and $\alpha(b)$ are finite, α is bounded on $[a,b]$.

Let P be a partition of $[a,b]$ and set
$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$
Then $\Delta\alpha_i \geq 0$.

Let f be a bounded real function on $[a,b]$.

Set $$U(P,f,\alpha) := \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$L(P,f,\alpha) := \sum_{i=1}^{n} m_i \Delta\alpha_i$$
(m_i, M_i defined as before)

Define $$\overline{\int_a^b} f\,d\alpha := \inf_{P\in[a,b]} U(P,f,\alpha)$$

& $$\underline{\int_a^b} f\,d\alpha := \sup_{P\in[a,b]} L(P,f,\alpha)$$

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote this common value by $\int_a^b f\,d\alpha$ or by $\int_a^b f(x)\,d\alpha(x)$. This is called the Riemann-Stieltjes integral, or simply the Stieltjes integral of f w.r.t. α, over $[a,b]$. We write $f \in \mathcal{R}(\alpha)$.

- $\alpha(x) = x$ gives the Riemann integral.
- However for a general α, even continuity is not required.

Q: When does $\int_a^b f\,d\alpha$ exist?

Defn. A partition P^* is said to be a refinement of P if $P^* \supset P$.

Given 2 partitions P_1 and P_2, we say P^* is their common refinement if $P^* = P_1 \cup P_2$.

Thm. 6.1 If P^* is a refinement of P, then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \quad \text{——} \quad (1)$$
$$U(P^*, f, \alpha) \leq U(P, f, \alpha) \quad \text{——} \quad (2)$$

Proof: We only prove (1). (2) can be similarly proved.

Suppose first that P^* contains only one more point than P, say x^*; let $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are 2 consecutive points of P.

Let $w_1 := \inf f(x) \quad (x_{i-1} \leq x \leq x^*)$
$w_2 := \inf f(x) \quad (x^* \leq x \leq x_i)$

Then $w_1 \geq m_i$ & $w_2 \geq m_i$, where
$m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i)$.

Thus,

$$\begin{aligned} & L(P^*, f, \alpha) - L(P, f, \alpha) \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i(\alpha(x_i) - \alpha(x_{i-1})) \\ &= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \\ &\geq 0. \end{aligned}$$

The case where P^* has more than 1 point than P can be handled similarly.

Thm. 6.2 $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$

Proof: Let P^* be a common refinement of P_1 & P_2. By the above theorem,

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$
$$\Rightarrow \quad L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Fix P_2 & take the supremum over all P_1 so that

$$\underline{\int_a^b} f\,d\alpha \leq U(P_2, f, \alpha).$$

Now take infimum over all P_2 so that

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

$\blacksquare$

Thm. 6.3 $f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$, $\exists$ a partition $P \ni U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Proof: "$\Leftarrow$" For every partition P,

$$L(P, f, \alpha) \leq \sup_{P' \in [a,b]} L(P', f, \alpha) = \underline{\int} f\,d\alpha \leq \overline{\int} f\,d\alpha$$
$$= \inf_{P' \in [a,b]} U(P', f, \alpha)$$
$$\leq U(P, f, \alpha).$$

Hence if $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$, then

$$0 \leq \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha < \varepsilon$$

Thus if $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ can be satisfied for every $\varepsilon > 0$, we have $\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha \Rightarrow f \in \mathcal{R}(\alpha)$.

$\underline{\int} f\, d\alpha = \inf_{P \in [a,b]} U(P, f, \alpha)$, $\overline{\int} f\, d\alpha = \sup_{P \in [a,b]} L(P, f, \alpha)$

"$\Rightarrow$" Suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$ is given. Then $\exists$ partitions P_1 and P_2 $\ni$

$$U(P_2, f, \alpha) < \int f\, d\alpha + \frac{\varepsilon}{2} \quad \text{— (a)}$$

$$\int f\, d\alpha - \frac{\varepsilon}{2} < L(P_1, f, \alpha) \quad \text{— (b)}$$

Now take P^* to be the common refinement of P_1 & P_2. By Thm. 6.1, (a) & (b),

$$U(P^*, f, \alpha) \leq U(P_2, f, \alpha) < \int f\, d\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon$$
$$\leq L(P^*, f, \alpha) + \varepsilon$$

$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \varepsilon$ for the partition P^*.

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