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MA 509 - REAL ANALYSIS - LECTURE 39

Thm. 6.4 (a) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ — $(*)$
for some P & some ε, then $(*)$ holds with the same ε for every refinement of P.

(b) If $(*)$ holds for $P = \{x_0, x_1, \cdots, x_n\}$ & if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$$

(c) If $f \in \mathcal{R}(\alpha)$ and the hypotheses of (b) holds, then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon.$$

Proof: (a) Let P' be a refinement of P. Then from Thm. 6.1 and Thm. 6.3,

$$U(P', f, \alpha) - L(P', f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

(b) From the hypotheses of (b), both $f(s_i)$ & $f(t_i)$ lie in $[m_i, M_i]$.

$$\Rightarrow |f(s_i) - f(t_i)| \leq M_i - m_i.$$

Thus,

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha)$$

$$< \varepsilon.$$

(c) Note that

$$L(P,f,\alpha) \le \sum f(t_i)\,\Delta\alpha_i \le U(P,f,\alpha)$$

& $$L(P,f,\alpha) \le \int f\,d\alpha \le U(P,f,\alpha)$$

Thus, $$\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha$$

$$\le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad \text{--- (1)}$$

Also, $$-\varepsilon < L(P,f,\alpha) - U(P,f,\alpha)$$

$$\le \sum f(t_i)\,\Delta\alpha_i - \int f\,d\alpha \quad \text{--- (2)}$$

From (1) & (2),

$$\left| \sum f(t_i)\,\Delta\alpha_i - \int f\,d\alpha \right| < \varepsilon.$$

$\boxtimes$

Thm. 6.5 If f is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Since α is monotonic, $\alpha(b) - \alpha(a) \ge 0$. Choose $\eta > 0 \ni (\alpha(b) - \alpha(a))\eta < \varepsilon$.

Since f is uniformly continuous on $[a,b]$, $\exists\, \delta > 0 \ni |f(x) - f(t)| < \eta$ whenever $x, t \in [a,b]$ & $|x - t| < \delta$.

If P is any partition of $[a,b]$ s.t. $\Delta x_i < \delta \ \forall i$, then from the above, $M_i - m_i \le \eta$ $(i = 1, 2, \ldots, n)$

Note that f is continuous on $[x_{i-1}, x_i]$ compact set so that f assumes its maximum & minimum, say, M_i & m_i respectively.

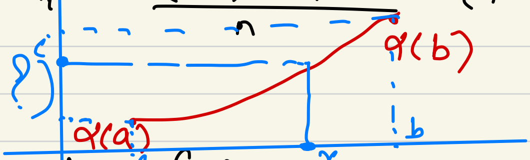
$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha)$$
$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \le \eta \sum_{i=1}^{n} \Delta\alpha_i$$
$$= \eta\,(\alpha(b) - \alpha(a))$$
$$< \varepsilon.$$

Hence by Thm. 6.3, $f \in \mathcal{R}(\alpha)$.

<u>Thm. 6.6</u> If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$.

<u>Proof:</u> Let $\varepsilon > 0$ be given. For any $n \in \mathbb{N}$, we choose a partition $P \ni \Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ $(i = 1, \cdots, n)$.

(Why is this possible?)



We prove the thm. when f is monotonically increasing. For f mon. decr., it can be proved similarly. Then
$$M_i = f(x_i), \quad m_i = f(x_{i-1}) \qquad (1 \le i \le n)$$
$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha)$$
$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$
$$= \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)) < \varepsilon,$$

since by Archimedean property, $\exists$ n can be chosen, in the beginning, to satisfy
$$n\varepsilon > (\alpha(b) - \alpha(a))(f(b) - f(a)).$$
By Thm. 6.3, $f \in \mathcal{R}(\alpha)$. $\blacksquare$

Thm. 6.7 Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, & α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

Remark: If f & α have a common point of discont-inuity, then f need not be in $\mathcal{R}(\alpha)$.

Thm. 6.8 Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

Which functions are Riemann integrable?

* Suppose f is bounded on $[a,b]$. Then $f \in \mathcal{R}$ iff f is continuous almost everywhere on $[a,b]$.

PROPERTIES OF THE INTEGRAL

Thm. 6.9 (a) If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf \in \mathcal{R}(\alpha)$ for any constant c, and

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha,$$

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha.$$

(b) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha.$$

(c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$, &

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha.$$

(d) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \leq M$ on $[a,b]$, then

$$\left| \int_a^b f\,d\alpha \right| \leq M(\alpha(b)-\alpha(a))$$

(e) If $f \in \mathcal{R}(\alpha_1)$ & $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1+\alpha_2)$ &

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2;$$

if $f \in \mathcal{R}(\alpha)$ and c is a positive constant, then $f \in \mathcal{R}(c\alpha)$ & $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$.