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MA 509 - REAL ANALYSIS - LECTURE 41

UNIFORM CONVERGENCE AND CONTINUITY

<u>Thm. 7.4</u> Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t \to x} f_n(t) = A_n \quad (n \in \mathbb{N})$.

Then $\{A_n\}$ converges, and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

In other words, $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$.

<u>Proof:</u> Let $\varepsilon > 0$ be given. Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N} \ni m, n \geq N, t \in E$ implies $|f_n(t) - f_m(t)| \leq \varepsilon$.

Let $t \to x$ and use the fact that $\lim_{t \to x} f_n(t) = A_n$ to conclude that $|A_n - A_m| \leq \varepsilon$ for $n, m \geq N$ so that $\{A_n\}$ is a Cauchy sequence. Since we concentrate on real-or complex-valued functions $\{A_n\}$ converges, say, to A.

Next,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \quad \text{---} (*)$$

Since $f_n \to f$ uniformly on E, choose $n \in \mathbb{N} \ni$ $|f(t) - f_n(t)| \leq \varepsilon/3 \quad \forall t \in E$, and such that $|A_n - A| \leq \varepsilon/3$.

(1)

(2)

Having chosen this n, choose nbhd V of $x \ni$ $|f_n(t) - A_n| \leq \frac{\varepsilon}{3}$ if $t \in V \cap E$, $t \neq x$.

(3)

$(\because \lim_{t \to x} f_n(t) = A_n)$

Substituting ①, ② & ③ in ⊛, we get

$|f(t) - A| \le \varepsilon$, provided $t \in V \cap E, t \ne x$.

$\Rightarrow \lim_{t \to x} f(t) = A$.

■

Thm. 7.5 If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Proof: Let x be any point of E. We are given that $f_n \to f$ uniformly on E, and moreover, f_n is continuous on E for each $n \in \mathbb{N}$.

$\Rightarrow \lim_{t \to x} f_n(t) = A_n \quad (n \in \mathbb{N})$.

Hence the hypotheses of the previous thm. are met.

$\Rightarrow \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$

$\Rightarrow \lim_{t \to x} f(t) = \lim_{n \to \infty} f_n(x) = f(x)$, so that f is continuous on x. Since x was an arbitrary point of E, we see that f is continuous on E.

The converse is not true! That is, $\exists$ a seq. of continuous functions converging to a continuous function, but the convergence is not uniform.

We have seen that if $f_n(x) := n^2 x (1 - x^2)^n$ on $[0, 1]$, then each f_n is continuous on $[0, 1]$, $f_n \to f$, where $f \equiv 0$ on $[0, 1]$, and hence, obviously, continuous. But the convergence is not uniform.

Thm. 7.6 (Dini's theorem)

Suppose K is compact and

(i) $\{f_n\}$ is a sequence of continuous functions on K

(ii) $\{f_n\}$ converges pointwise to a continuous fn. f on K

(iii) $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K$ & $n \in \mathbb{N}$.

Then $f_n \to f$ uniformly on K.

Proof: Let $g_n = f_n - f$. Obviously, g_n is continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$.

Claim: $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let

$$K_n = \{x \in K : g_n(x) \geq \varepsilon\}.$$

Then g_n is continuous and K_n is closed. (Why?)

Inverse image of closed set under a continuous map is closed. Since $[\varepsilon, \infty)$ is closed, so is K_n as g_n is continuous on K.

$\Rightarrow K_n$ is compact (Why?)

K_n is closed, $K_n \subset K$, where K is compact.

$\Rightarrow K_n$ is compact.

Since $g_n \geq g_{n+1}$, $K_n \supset K_{n+1}$.

Fix $x \in K$. Since $g_n(x) \to 0$ (pointwise), we have $x \notin K_n$ if n is sufficiently large.

$\Rightarrow x \notin \bigcap K_n$ so that $\bigcap K_n$ is empty.

$\Rightarrow K_N = \phi$ for some $N \in \mathbb{N}$

Thus $\forall x \in K$, and for all $n \in \mathbb{N}$, we must have $0 \leq g_n(x) < \varepsilon$.

$\Rightarrow g_n \to 0$ uniformly on K. ▨

Defn. If X is a metric space, $\mathscr{C}(X)$ denotes the set of all complex-valued, continuous, bounded functions on X.

We associate with each $f \in \mathscr{C}(X)$ its <u>supremum norm</u> $\|f\| := \sup_{x \in X} |f(x)|$.

Since f is bounded on X, $\|f\| < \infty$.
Clearly, $\|f\| = 0$ iff $f(x) = 0 \ \forall x \in X$, i.e. $f \equiv 0$.
If $h = f + g$, then
$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\| \quad \forall x \in X$
$\Longrightarrow \|f+g\| \leq \|f\| + \|g\|$.

Define the distance between $f, g \in \mathscr{C}(X)$ to be $\|f-g\|$. Then from the above, we see that $\mathscr{C}(X)$ is a metric space.

Remark:
Thus Thm. 7.2 can be rephrased as:

A sequence $\{f_n\}$ converges to f w.r.t. the metric of $\mathscr{C}(X)$ iff $f_n \to f$ uniformly on X.

Hence, closed subsets of $\mathscr{C}(X)$ are called uniformly closed, and the closure of $A \in \mathscr{C}(X)$, the uniform closure.

Thm. 7.7 The metric defined above makes $\mathscr{C}(X)$ into a complete metric space.