25/8/2020 MA 509 - (Lecture 4) REAL ANALYSIS

Thm. 1.3 The field axioms imply the following stmts. for any $x, y, z \in F$.

(a) $0x = 0$

(b) If $x \neq 0$ & $y \neq 0$, then $xy = 0$.

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

Proof: (a) $0x + 0x = (0+0)x = 0x$. By Thm. 1.2(b), $0x = 0$.

(b) Assume $x \neq 0, y \neq 0$, but $xy = 0$. Then

$$1 = \left(\frac{1}{y} \cdot y\right)\left(\frac{1}{x} \cdot x\right) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right) \cdot (xy) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right) \cdot 0 = 0,$$

a contradiction.

(c) $(-x)y + xy = (-x+x)y = 0 \cdot y = 0$

$\Rightarrow (-x)y = -(xy)$.

Similarly, the second part.

(d) Now by (c) and Thm. 1.2 (d),

$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$.

ORDERED FIELD

An ordered field is a field F which is also an ordered set s.t.

(i) $x + y < x + z$, if $x, y, z \in F$ and $y < z$.

(ii) $xy > 0$ if $x \in F, y \in F, x > 0, y > 0$.

e.g. $\mathbb{Q}$ is an ordered field.

Thm. 1.4 The following stmts. are true in every ordered field

(a) If $x>0$, then $-x<0$, and vice-versa
(b) If $x>0$ and $y<z$, then $xy<xz$
(c) If $x<0$ and $y<z$, then $xy>xz$.
(d) If $x\neq 0$, then $x^2>0$. In particular, $1>0$.
(e) If $0<x<y$, then $0<1/y<1/x$.

Proof: (a) If $x>0$, then $0=-x+x>-x+0=-x$.

$\Rightarrow -x<0$.

If $x<0$, then $0=-x+x<-x+0=-x \Rightarrow -x>0$.

(b) Since $z>y$, $z-y>y-y=0$. $\Rightarrow x(z-y)>0$

$\Rightarrow xz=x(z-y)+xy>0+xy=xy$.

(c) By (a), (b) and Thm. 1.3 (c),

$-(x(z-y))=(-x)(z-y)>0$

$\Rightarrow x(z-y)<0 \Rightarrow xz<xy$.

(d) If $x>0$, then $x^2>0$. If $x<0$, then $-x<0 \Rightarrow (-x)^2>0$.
But $x^2=(-x)^2$ by Thm. 1.3 (d). Hence $x^2>0$ for $x\neq 0$.
(as $x=0 \Rightarrow x^2=0$).

Since $1^2>0$ & $1=1^2$, we have $1>0$.

(e) If $y>0$ and $v\leq 0$, then $yv\leq 0$.

So $\frac{1}{y}>0$, as $y\cdot(1/y)=1>0$.

Similarly, $1/x>0$. Now if $x<y$ implies

$\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\cdot x<\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\cdot y \Rightarrow \frac{1}{y}<\frac{1}{x}$. ▢

THE REAL FIELD

Thm. 1.4 There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property.

Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield; $\mathbb{Q} \subseteq \mathbb{R}$.

(known as the set of real numbers)

Thm. 1.5

(a) ARCHIMEDEAN PROPERTY

If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, $\exists\, n \in \mathbb{N} \ni nx > y$.

(b) If $x, y \in \mathbb{R}$ and $x < y$, $\exists\, p \in \mathbb{Q} \ni x < p < y$.

(This implies $\mathbb{Q}$ is dense in $\mathbb{R}$)

Proof: (a) (by contradiction)

Suppose $\nexists\, n \in \mathbb{N} \ni nx > y$. Then $nx \leq y \;\forall\, n \in \mathbb{N}$.

Hence if $A = \{nx : n \in \mathbb{N}\}$, then y is an upper bound of A. (Note that $x \in A$, hence $A \neq \phi$.)

By the lub property, A has the least upper bound in $\mathbb{R}$, say α.

Now since $x > 0$, $\alpha - x < \alpha$, and then $\alpha - x$ is not an upper bound of A.

$\Rightarrow \exists\, m \in \mathbb{N} \ni \alpha - x < mx$.

$$\Rightarrow \alpha < (m+1)x.$$

This contradicts the fact that α is an upper bound of A. This establishes the claim. $\boxtimes$

(b) Since $x<y$, $y-x>0$; hence by Archimedean property, $\exists\, n \in \mathbb{N} \ni n(y-x)>1$ ——— (1)

Again, by (a), $\exists\, m_1, m_2 \in \mathbb{N} \ni m_1 > nx$ & $m_2 > -nx$.

$\Rightarrow \quad -m_2 < nx < m_1$

Hence $\exists\, m \in \mathbb{Z}$ with $-m_2 \leq m \leq m_1$ s.t.

$$m-1 \leq nx < m \qquad \text{——— (2)}$$

From (1) & (2),

$$nx < m \leq nx+1 < ny$$

Since $n>0$,

$$x < \frac{m}{n} < y$$

$\Rightarrow$ the rational number m/n lies between x & y. ▨

EXISTENCE OF n^{TH} ROOTS OF POSITIVE REALS

Thm 1.6 For every real $x>0$ & every integer $n>0$, $\exists!$ real $y>0 \ni y^n = x$.

(there exists a unique)

(This number y is written as $x^{1/n}$ or $\sqrt[n]{x}$.)

Proof: The uniqueness is clear, for, if $0<y_1<y_2$ with $y_1^n = y_2^n = x$, we have $y_1^n < y_2^n$, which is absurd.

Let $E = \{t \in \mathbb{R}^+ : t^n < x\}$.

We first show that E is non-empty & bdd. above.

Note that if $t = \frac{x}{1+x}$, then $0<t<1$ ($\because x>0$).

Hence, $t^n < t < x$. $\Rightarrow \frac{x}{1+x} \in E$, so $E \neq \phi$.

Moreover, if $t' > 1+x$, then $t'^n > t' > x$, so $t' \notin E$

By contrapositivity, if $t' \in E$, then $t' \leq 1+x$

$\Rightarrow 1+x$ is an upper bound of E

By lub property, E has the least upper bound in $\mathbb{R}$, say $y = \sup(E)$.

<u>Claim:</u> $y^n = x$.

This is done by showing $y^n \not< x$ and $y^n \not> x$.

Note first that for $0 < a < b$,

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + b a^{n-2} + a^{n-1})$$
$$< (b-a) n b^{n-1}. \qquad \text{———} \; (*)$$

Assume $y^n < x$. Choose $h \ni 0 < h < 1$ &

$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$

Let $a = y$, $b = y+h$ in $(*)$. Then

$$(y+h)^n - y^n < h n (y+h)^{n-1} < h n (y+1)^{n-1} < x - y^n$$

$\Rightarrow (y+h)^n < x$ and so $y + h \in E$

But $y + h > y = \sup(E)$. This contradicts the fact that y is an upper bound of E. $\Rightarrow y^n \not< x$.

Now assume $y^n > x$. Let $k = \dfrac{y^n - x}{n y^{n-1}}$.

Note that $x > 0 \Rightarrow -x < 0 \Rightarrow y^n - x < y^n \leq n y^n$

Since $y > 0$, $\dfrac{y^n - x}{n y^{n-1}} \leq y$, i.e., $k < y$.

So $0 < k < y$. If $\underline{t \geq y - k}$, then

$$y^n - t^n \leq y^n - (y-k)^n < k n y^{n-1} = y^n - x.$$

$\Rightarrow t^n > x$ and so $t \notin E$.

Hence $y - k$ is an upper bound of E. But $y - k < y$ ($\because k > 0$). This contradicts the fact that y is the least upper bound of E.

Hence $y^n \not> x$. Thus, we conclude that $y^n = x$. ∎