(b) Since $x<y$, $y-x>0$; hence by Archimedean property, $\exists\, n\in\mathbb{N} \ni n(y-x)>1$ ——— (1)

Again, by (a), $\exists\, m_1, m_2 \in \mathbb{N} \ni m_1 > nx$ & $m_2 > -nx$.

$$\Rightarrow -m_2 < nx < m_1$$

Hence $\exists\, m\in\mathbb{Z}$ with $-m_2 \le m \le m_1$ s.t.

$$m-1 \le nx < m \quad \text{——— (2)}$$

From (1) & (2),

$$nx < m \le nx+1 < ny$$

Since $n>0$,

$$x < \frac{m}{n} < y$$

$\Rightarrow$ the rational number m/n lies between x & y. ∎

EXISTENCE OF n^{TH} ROOTS OF POSITIVE REALS

Thm 1.6 For every real $x>0$ & every integer $n>0$, $\exists!$ real $y>0 \ni y^n = x$.

(there exists a unique) (This number y is written as $x^{1/n}$ or $\sqrt[n]{x}$.)

Proof: The uniqueness is clear, for, if $0<y_1<y_2$ with $y_1^n = y_2^n = x$, we have $y_1^n < y_2^n$, which is absurd.

Let $E = \{t\in\mathbb{R}^+ : t^n < x\}$.

We first show that E is non-empty & bdd. above.

Note that if $t = \frac{x}{1+x}$, then $0<t<1$ ($\because x>0$).

Hence, $t^n < t < x$. $\Rightarrow \frac{x}{1+x} \in E$, so $E \neq \phi$.

Moreover, if $t' > 1+x$, then $t'^n > t' > x$, so $t' \notin E$

By contrapositivity, if $t'\in E$, then $t' \le 1+x$

$\Rightarrow 1+x$ is an upper bound of E

By lub property, E has the least upper bound in $\mathbb{R}$, say $y = \sup(E)$.

<u>Claim</u>: $y^n = x$.

This is done by showing $y^n \not< x$ and $y^n \not> x$.

Note first that for $0 < a < b$,

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$$
$$< (b-a)\, n\, b^{n-1}. \quad \text{——} \ (*)$$

Assume $y^n < x$. Choose $h \ni 0 < h < 1$ &

$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$

Let $a = y$, $b = y+h$ in $(*)$. Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

$\Rightarrow (y+h)^n < x$ and so $y+h \in E$.

But $y+h > y = \sup(E)$. This contradicts the fact that y is an upper bound of E. $\Rightarrow y^n \not< x$.

Now assume $y^n > x$. Let $k = \dfrac{y^n - x}{n y^{n-1}}$.

Note that $x > 0 \Rightarrow -x < 0 \Rightarrow y^n - x < y^n \leq n y^n$

Since $y > 0$, $\dfrac{y^n - x}{n y^{n-1}} < y$, i.e., $k < y$.

So $0 < k < y$. If $\underline{t \geq y - k}$, then

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

$\Rightarrow t^n > x$ and so $t \notin E$.

Hence $y-k$ is an upper bound of E. But $y - k < y$ ($\because k > 0$).

This contradicts the fact that y is the least upper bound of E.

Hence $y^n \not> x$. Thus, we conclude that $y^n = x$. ∎

Cor. 1.7 If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof: Let $\alpha = a^{1/n}$ and $\beta = b^{1/n}$. Then

$$ab = \underbrace{\alpha^n \beta^n = (\alpha\beta)^n}_{\text{by commutativity}}.$$

$$\xRightarrow[\text{uniqueness}]{\text{by}} \alpha\beta = (ab)^{1/n}$$

$$\Rightarrow (ab)^{1/n} = a^{1/n} b^{1/n}. \quad \blacksquare$$

RELATION BETWEEN REAL NUMBERS AND DECIMALS

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$. (How do we know such an n_0 exists?)

By induction, having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer s.t.

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Let $E = \left\{ n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} : k \in \mathbb{N} \cup \{0\} \right\}$

Claim: $x = \sup(E)$ (Why?)

The decimal expansion of x is $n_0 . n_1 n_2 n_3 \ldots$

Conversely, for any infinite decimal (6) the set E of numbers (5) is bounded above, and (6) is the decimal expansion of $\sup(E)$.