ATUL DIXIT

8] Let f be a real uniformly continuous function on a bounded set E in R^1. Prove that f is bounded on E.
Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof :- First we prove one lemma :-

Lemma :- Suppose (M_1, ρ_1) is a metric space and A is a dense subset of M_1. If f is a uniformly continuous function from (A, ρ_1) into a complete metric space (M_2, ρ_2), then f can be extended to a uniformly continuous function F from M_1 into M_2.

Proof: First we will prove that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of points in A, $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy in M_2.
Indeed because of uniform continuity of f on A, given $\varepsilon > 0$, $\exists\ \delta > 0 \ni$

$$\rho_2(f(x), f(y)) < \varepsilon \quad \text{whenever } \rho_1(x,y) < \delta$$
$$(\text{where } x, y \in A) \quad -(i)$$

Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in A, there exists $N \in \mathbb{N} \ni \rho_1(x_m, x_n) < \delta$ (where $m, n \geq N$) $-(ii)$

Thus $(i), (ii)$ imply that

$$\rho_2(f(x_m), f(x_n)) < \varepsilon \quad (m, n \geq N)$$

Thus $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy sequence in M_2

Now we define Function F as :-

$$F(x) = f(x) \quad \forall\ x \in A.$$

Also if $x \in M_1$ and $x \notin A$, then since A is dense in M_1 (i.e. $\bar{A} = M_1$), $x \in \bar{A} - A = A'$.
Hence x is a limit point of A.
Thus $\exists$ sequence $\{x_n\}_{n=1}^{\infty}$ of points of $A \ni$
$\lim_{n \to \infty} x_n = x$.

So since $\{x_n\}_{n=1}^{\infty}$ is convergent sequence in a metric space (M_1, ρ_1), $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy sequence in M_2.
Also since M_2 is complete metric space, this Cauchy sequence converges to some point in M_2.

So for such $x \in M_1$ and $x \notin A$, define

$$F(x) = \lim_{n \to \infty} f(x_n)$$

Also since f being uniformly continuous is continuous, if there is a sequence $\{x_n\} \ni x_n \to x$, then $f(x_n) \to f(x)$
~~Thus~~ If there is another sequence $\{y_n\} \ni$ $y_n \to x$, then ~~$f(y_n) \to f(x)$~~
~~Thus~~ $\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} f(x_n)$

Thus this definition of $F(x)$ is independent of the choice of sequence $\{x_n\}_{n=1}^{\infty}$

Thus $F(x)$ is defined for all $x \in M_1$ and it is clearly ~~an~~ extension of f.

To prove that F is uniformly continuous:-

Given $\varepsilon > 0$, $\exists\, \delta_1 > 0 \ni$

$\rho_2(f(x), f(y)) < \varepsilon/3$ — (A) whenever $\rho_1(x,y) < \delta_1$ where $x, y \in A$

If $a, b \in M_1$, choose $x, y \in A \ni \rho_1(x,a) < \delta_1/3$

$\rho_1(y,b) < \delta_1/3$, and such that

$\rho_2(F(x), F(a)) < \varepsilon/3$ — B

and $\rho_2(F(y), F(b)) < \varepsilon/3$ — C

(This is possible because since $F(x) = f(x)$ and $F(a) = \lim_{n\to\infty} f(x_n)$ for some sequence $\{x_n\}$ converging to a, if x is within $\delta_1/3$ distance from a, then its possible to have their images under F within $\varepsilon/3$ distance.)

Now if $\rho_1(a,b) < \delta_1/3$, by triangle inequality,

$$\rho_1(x,y) \le \rho_1(x,a) + \rho_1(a,b) + \rho_1(b,y) < \frac{\delta_1}{3} + \frac{\delta_1}{3} + \frac{\delta_1}{3} = \delta_1$$

Thus $\rho_1(x,y) < \delta_1$

But then

$$\rho_2(F(x), F(y)) = \rho_1(f(x), f(y)) < \varepsilon/3$$

(from A)

Thus from A, B, C, and by triangle inequality,

$$\rho_2(F(a), F(b)) < \rho_2(F(a), F(x)) + \rho_2(F(x), F(y)) + \rho_2(F(y), F(b)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\therefore \rho_2(F(a), F(b)) < \varepsilon$ whenever $\rho_1(a,b) < \delta_1/3$

This implies that F is uniformly continuous on M_1.

Hence proved.

Now if in the statement of this lemma, we replace M_1 by $\bar{E}$, A by E (we can do so because E is dense in $\bar{E}$, since $\bar{E} = \bar{E}$) and M_2 by $\mathbb{R}$ ($\mathbb{R}$ is a complete metric space since every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$), then we can extend the given real valued uniformly continuous function on bounded set E (in R^1) to a uniformly continuous function F from $\bar{E}$ into $\mathbb{R}$.

— I

Now we prove the given theorem :-

Since E is bounded in R^1,

$\text{diam}(E) < \infty$

But $\text{diam}(E) = \text{diam}(\bar{E})$

$\therefore \text{diam}(\bar{E}) < \infty$

This means that $\bar{E}$ is bounded in R^1

But $\bar{E}$ is also closed in R^1.

Hence by Heine-Borel theorem, $\bar{E}$ is compact.

Now from ~~$\ast$~~ I, since F is uniformly continuous on $\bar{E}$, it is also continuous on $\bar{E}$ and also $\bar{E}$ is compact.

This means that $F(\bar{E})$ is compact (since continuous image of a compact set is compact)

But since $F(\bar{E}) \subset \mathbb{R}^1$, by Heine-Borel theorem $F(\bar{E})$ is closed and bounded.

But this means that F is bounded on $\bar{E}$.
So it is also bounded on E.
But on E, F and f are identical
(from I)

Therefore f is bounded on E.

Hence proved.

If E is not bounded, as in the case when $E = \mathbb{R}^+$, then take $f(x) = \sqrt{x}$ $\forall\, x \in \mathbb{R}^+$. f is a real valued uniformly continuous fn. on $\mathbb{R}^+$, but $f(\mathbb{R}^+)$ is not bounded.
(because as $x \to \infty$, $f(x) \to \infty$.)

① Let f be defined for all real x, and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all real x and y. Prove that f is constant.

Proof :- $|f(x) - f(y)| \leq (x-y)^2 \quad \forall\ x, y \in \mathbb{R}$.

Now $(x-y)^2 \geq 0$. So for $x \neq y$, $0 < (x-y)^2 = |x-y|^2$.

$$\therefore \frac{|f(x) - f(y)|}{(x-y)^2} \leq 1$$

$$\therefore \left| \frac{f(x) - f(y)}{(x-y)^2} \right| \leq 1$$

$$\therefore 0 \leq \left| \frac{f(x) - f(y)}{x-y} \right| \leq |x-y|$$

Taking $\lim\limits_{x \to y}$ on both sides,

$$0 \leq \lim_{x \to y} \left| \frac{f(x) - f(y)}{x-y} \right| \leq \lim_{x \to y} |x-y|$$

$$\therefore 0 \leq \left| \lim_{x \to y} \frac{f(x) - f(y)}{x-y} \right| \leq 0$$

$$\Rightarrow \left| \lim_{x \to y} \frac{f(x) - f(y)}{x-y} \right| = 0 \quad \Rightarrow \lim_{x \to y} \frac{f(x) - f(y)}{x-y} = 0.$$

$\cancel{\therefore |f'(y)| \leq 0}$

~~But absolute value of any number is either positive or zero.~~

Hence f is differentiable at y (and hence on $\mathbb{R}$).

~~Hence~~ $\cancel{|f'(y)|} = 0$ & $f'(y) = 0$.

i.e. $f'(y) = 0$ and this true $\forall\ y \in \mathbb{R}$

$\therefore f$ is constant. Hence proved

2) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$ $(a < x < b)$

Proof :- Let $x_1 < x_2$ where $x_1, x_2 \in (a, b)$

Now since $(x_1, x_2) \subset (a, b)$, f is also differentiable on (x_1, x_2)

Hence by Mean value theorem, there exists $t \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(t)\,(x_2 - x_1)$$

Now since $x_1 < x_2$, $x_2 - x_1 > 0$

Also $f'(t) > 0$ where $t \in (a, b)$

$\therefore$ $f(x_2) - f(x_1) > 0$

i.e. $f(x_1) < f(x_2)$

Hence f is strictly increasing in (a, b).

Hence proved.

Let $g = f^{-1}$. Let $p, q \in (a, b) \ni p = f^{-1}(x)$ and $q = f^{-1}(y)$ for $x, y \in f((a,b))$

Then $\lim\limits_{x \to y} \dfrac{g(x) - g(y)}{x - y}$

$$= \lim_{x \to y} \frac{f^{-1}(x) - f^{-1}(y)}{x - y}$$

$$= \lim_{x \to y} \frac{p - q}{f(p) - f(q)}$$

$$= \lim_{f(p) \to f(q)} \frac{p - q}{f(p) - f(q)}$$

Now since f is differentiable in (a, b), f is continuous, so if $f(p) \to f(q)$, we must have $p \to q$ (because if $p \to q'$, then we must ~~have~~ have by continuity, that $f(p) \to f(q')$; But then $\{f(p)\}$ cannot converge to ~~a~~ $f(q)$ as well as $f(q')$ in $\mathbb{R}$. ~~Hence~~ ~~$f(q) = f(q')$ i.e.~~ which is a contradiction.

$$\therefore \lim_{x \to y} \frac{g(x) - g(y)}{x - y}$$

$$= \lim_{p \to q} \frac{p - q}{f(p) - f(q)}$$

$$= \lim_{p \to q} \frac{1}{\left(\frac{f(p) - f(q)}{p - q} \right)}$$

$$= \frac{1}{\lim_{p \to q} \frac{f(p) - f(q)}{p - q}}$$

$$= \frac{1}{f'(q)}$$

$(\because f'(x) > 0 \ \forall x \in (a, b))$

Hence $1/f'(q)$ exists.

$\therefore \lim_{x \to y} \frac{g(x) - g(y)}{x - y}$ exists.

Thus g is differentiable at y.
Similarly g is differentiable at other points belonging to (a, b)

Now $(g \circ f)(x) = g(f(x))$
$= f^{-1}(f(x))$
$= x$

$\therefore (g \circ f)(x) = x$
Now since $(g \circ f)$ is differentiable at x (by thm. 5.5),
$(g \circ f)'(x) = 1$
$\therefore g'(f(x)) \cdot f'(x) = 1$
$\therefore g'(f(x)) = \frac{1}{f'(x)}$

Hence proved.

3) Suppose g is a real function of R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$ and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

Proof :- ~~Let $x \neq y$, we will prove that $f(x) \neq f(y)$,~~

Now $|g'(y)| \leq M$

$$\therefore \left| \lim_{x \to y} \frac{g(x) - g(y)}{x - y} \right| \leq M$$

$\because \varepsilon > 0$, multiplying both sides by ε,

$$\therefore \left| \lim_{x \to y} \left\{ \frac{\varepsilon g(x) - \varepsilon g(y)}{x - y} \right\} \right| \leq M\varepsilon$$

$$\therefore \left| \lim_{x \to y} \frac{x + \varepsilon g(x) - (y + \varepsilon g(y))}{x - y} - 1 \right| \leq M\varepsilon$$

$$\therefore \left| \lim_{x \to y} \frac{f(x) - f(y)}{x - y} - 1 \right| \leq M\varepsilon$$

Now since f is a addition of 2 differentiable ~~continuous~~ functions $f_1(x) = x$ & $f_2(x) = \varepsilon g(x)$, f is ~~continuous~~ differentiable

$\therefore \lim_{x \to y} \frac{f(x) - f(y)}{x - y}$ exists and is equal to

$f'(y)$

$\therefore |f'(y) - 1| \leq M\varepsilon$

$\Rightarrow \quad -M\varepsilon \leq f'(y) - 1 \leq M\varepsilon$

$\Rightarrow \quad f'(y) \geq 1 - M\varepsilon$, in particular

Now if ε is small enough, $1 - M\varepsilon > 0$; this precisely happens when $\varepsilon < 1/M$.

$\Rightarrow \quad f'(y) > 0$ for $\varepsilon < 1/M$.

Then by the previous problem, f is one-one for the admissible values of ε, depending only on M.

$\blacksquare$

Hence proved.

4] If $c_0 + \frac{c_1}{2} + \dots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0$, where $c_0, c_1, \dots, c_n$ are real constants, prove that the equation $c_0 + c_1 x + \dots + c_{n-1}x^{n-1} + c_n x^n = 0$ has atleast one real root between 0 and 1

Proof :- Construct a new function g such that

$$g(x) = \sum_{i=0}^{n} \frac{c_i x^{i+1}}{i+1}$$

Now $g(x)$ being a polynomial in x, is differentiable. So it is also differentiable in $(0,1)$ — (1)

Now $g(0) = \sum_{i=0}^{n} \frac{c_i}{i+1} \cdot (0)^{i+1} = 0$

Also $g(1) = \sum_{i=0}^{n} \frac{c_i}{i+1} (1)^{i+1} = 0$ (given)

$\therefore g(0) = g(1)$ — (a)

Now since g is differentiable, g is also continuous. So f is continuous, in $[0,1]$ — (2)

Now from (1) and (2), and by Mean value theorem there exists an $p \in [0,1] \ni$

$g'(p) = \frac{g(1) - g(0)}{(1-0)}$ — (b)

$\therefore$ From (a) and (b),

$g'(p) = 0$

But $g'(x) = \sum_{i=0}^{n} \frac{(i+1) c_i x^i}{(i+1)} = \sum_{i=0}^{n} c_i x^i$

Hence ~~$\sum_{i=0}^{n} c_i x^i = 0$ for this~~

Hence $g'(p) = \sum_{i=0}^{n} c_i p^i = 0$.

Hence $p \in (0,1)$ is the root of $\sum_{i=0}^{n} c_i x^i = 0$.

Hence proved.