27/11/2020

MA 509 - Tutorial 12 solutions

① Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof :- Let $\{f_n\}$ be the uniformly convergent sequence s.t. $|f_n(x)| \leq M_n$ for every $n \in \mathbb{N}$ & $\forall x$.

Since the Cauchy criterion for unif. conv. holds, $\exists N \in \mathbb{N}$ such that for $\forall x$, & $m \geq N$,

$$|f_m(x) - f_N(x)| \leq 1.$$

Then for all such m,

$$\begin{aligned} |f_m(x)| &= |f_m(x) - f_N(x) + f_N(x)| \\ &\leq |f_m(x) - f_N(x)| + |f_N(x)| \\ &\leq 1 + M_N. \end{aligned}$$

Then let $M = \max(M_1, M_2, \ldots, 1 + M_N)$.

$\Rightarrow |f_n(x)| \leq M \quad \forall n \in \mathbb{N}$ & $\forall x$.

$\boxtimes$

① $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$

ⓐ Claim: The series converges absolutely for all x except $x=0$ and $x = -1/n^2$, $n \in \mathbb{N}$

Obviously, $x=0$ & $x = -1/n^2$, $n \in \mathbb{N}$ have to be excluded.

So consider $x \neq 0$ & $x \neq -1/n^2$, $n \in \mathbb{N}$:

① Case 1: Let $x<0$. By reverse-Δ inequality $|1+n^2x| \geq \big||1| - n^2|x|\big|$, so, in particular,

$|1+n^2x| \geq 1 - n^2|x| > -n^2|x|$

$\Rightarrow \frac{1}{|1+n^2x|} < \frac{-1}{n^2|x|}$ (Note that $|x| = -x$ since $x<0$, so RHS is positive)

Thus $\sum_{n=1}^{\infty} \frac{1}{|1+n^2x|} < \frac{-1}{|x|} \sum_{n=1}^{\infty} \frac{1}{n^2}$.

② Case 2: Let $x>0$. Then $|1+n^2x| = 1+n^2x > n^2x$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{|1+n^2x|} < \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

In both the case, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test,

$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges absolutely for $\forall x \neq 0, -1/n^2$, $n \in \mathbb{N}$.

ⓑ It converges uniformly on $\mathbb{R} \setminus (-\delta, \delta)$ for any $\delta > 0$, except for $x = -1/n^2$, $n \in \mathbb{N}$.

(i) Let $x \in [\delta, \infty)$. Then

$|1+n^2x| = 1+n^2x > 1+n^2\delta > n^2\delta$ so that

$$\left|\frac{1}{1+n^2x}\right| < \frac{1}{\delta n^2}$$

Hence by Weierstrass-M test, $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges uniformly when $x\in[\delta,\infty)$.

(ii) Now let $x\in(-\infty,-\delta]$. Consider $n\geq\sqrt{2/\delta}$.

Then $$\frac{1}{|1+n^2x|} = \frac{1}{n^2\left|x+\frac{1}{n^2}\right|}.$$

Now $\left|x+\frac{1}{n^2}\right| \geq \left||x|-\frac{1}{n^2}\right|$, i.e., in particular,

$$\left|x+\frac{1}{n^2}\right| \geq |x|-\frac{1}{n^2} = -x-\frac{1}{n^2} \geq \delta-\frac{1}{n^2}\geq\frac{\delta}{2}$$

$$\left(\text{since } n^2\geq\frac{2}{\delta} \Leftrightarrow \frac{1}{n^2}\leq\frac{\delta}{2} \Leftrightarrow -\frac{1}{n^2}\geq-\frac{\delta}{2} \Leftrightarrow \delta-\frac{1}{n^2}\geq\frac{\delta}{2}\right)$$

$$\Rightarrow \frac{1}{|1+n^2x|} \leq \frac{2}{n^2\delta}$$

Then by Weierstrass-M test, $\sum\frac{1}{1+n^2x}$ converges uniformly on $(-\infty,-\delta]$.

(c) The series does not converge uniformly on $[0,\delta)$ or on $(-\delta,0]$ for any $\delta>0$.

This is because if the series converged uniformly on $[0,\delta)$, then, from prob. (1), the sequence of partial sums of the series is uniformly bounded, since the partial sums are bounded. (to see this, note that

$$\sum_{n=1}^{k} \frac{1}{1+n^2 x} < \sum_{n=1}^{k} 1 = k \,.\Big)$$

But then the limit function, i.e., $f(x)$ must be bounded. However,

$$\left.\begin{aligned} f\left(\frac{1}{m^2}\right) &= \sum_{n=1}^{\infty} \frac{1}{1+\frac{n^2}{m^2}} \geq \sum_{n=1}^{m} \frac{1}{1+\frac{n^2}{m^2}} \\ &\geq \sum_{n=1}^{m} \frac{1}{2} \quad (\because n \leq m) \\ &= \frac{m}{2}\,. \end{aligned}\right\} \quad (*)$$

This shows f is unbounded. $\rightarrow\!\leftarrow$.
Hence the series does not converge uniformly on $[0, \delta)$ for any $\delta > 0$.

Now if the series converged uniformly on $(-\delta, 0]$, then the sequence of its partial sums satisfies the Cauchy criterion for uniform convergence.

However, at $x = 1/m^2$, for any m,

$$2 = \frac{1}{1+m^2 x} = \sum_{n=1}^{m} \frac{1}{1+n^2 x} - \sum_{n=1}^{m-1} \frac{1}{1+n^2 x}.$$

$\rightarrow\!\leftarrow$.

(d) Wherever the series converges uniformly, i.e.; in this case, $\mathbb{R} \setminus (-\delta, \delta)$ for any $\delta > 0$, and wherever it is defined, i.e. excluding the pts. $x = -1/m^2$, $m \in \mathbb{N}$, the function f is continuous because it is a limit of unif. conv. seq. of cont. fns.

(c) From (*), we see that f is not bounded.