MA 509: Tutorial 3 (2020)

1. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

2. If x, y are complex, prove that

$$|\ |x| - |y|\ | \leq |x - y|.$$

3. Prove that

$$|\ \mathbf{x} + \mathbf{y}\ |^2 + |\ \mathbf{x} - \mathbf{y}\ |^2 = 2\ |\ \mathbf{x}\ |^2 + 2\ |\ \mathbf{y}\ |^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

4. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

5. A complex number z is said to be *algebraic* if there are integers $a_0, a_1, \cdots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdot + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers if countable.

(Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.)$$