Rudin : Chap. 2

(6) Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. Do E and E' always have the same limit points?

(a) To prove that E' is closed. We show that E'^c is open.

(i) Suppose $x \in E'^c$ and $x \notin E$. Then $x \notin \bar{E}$ and hence $x \in \bar{E}^c$. But from Thm. 2.27, $\bar{E}$ is closed, or in other words $\bar{E}^c$ is open. So $\exists$ a nbhd N of $x \ni N \subset \bar{E}^c$.

But $E' \subset \bar{E} \Rightarrow E'^c \supset \bar{E}^c$. So $N \subset E'^c$.

$\Rightarrow x$ is an interior point of E'^c.

(ii) Suppose $x \in E'^c$ and $x \in E$. Then x is an isolated point of E. So $\exists$ nbhd M of $x \ni M \cap E = \{x\}$. We show that $M \cap E' = \phi$, which will again imply that x is an interior point of E'^c. To that end, suppose by contradiction, $M \cap E' \neq \phi$ and let $y \in M \cap E'$. Then y being a limit point of E, by Thm. 2.20, every neighborhood of y contains infinitely many points of E. So choose a nhbd of y which lies completely in M. This implies that M contains infinitely many points of E, which contradicts the fact that $M \cap E = \{x\}$.

Thus from (i) and (ii), every point $x \in E'^c$ is an interior point of E'^c. Thus E'^c is open.

$\blacksquare$

(b) Prove that E and $\bar{E}$ have the same limit points.

Proof:- We want to show that $E' = \bar{E}'$.

(i) $\bar{E}' \subset E'$

Proof:- From (a) E' is closed, so it contains all limit points of E'. But by defn, E' contains all limit points of E. So E' contains all limit points of $E \cup E' = \overline{E}$.

$\Rightarrow \overline{E}' \subset E'$

(ii) $E' \subset \overline{E}'$.

Proof:- Suppose x is a limit point of E, but not of $\overline{E}$. Then $\exists$ nbhd N of $x \ni N \cap \overline{E} = \{x\}$. But x being a limit point of E, every nbhd of x, in particular N, contains infinitely many points of E (and hence of $\overline{E}$). This is a contradiction. Thus if $x \in E'$, $x \in \overline{E}'$.

$\Rightarrow E' \subset \overline{E}'$

From (i) and (ii), $E' = \overline{E}'$. So E and $\overline{E}$ always have same limit points.

(c) No E and E' do not always have same limits. For e.g., let $E = \{\frac{1}{n} : n \in \mathbb{N}, n > 1\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}, n > 1\}$.

Then $E' = \{0, 1\}$, but $E'' = \phi$, so $E' \neq E''$.

(2) Prove that the set of all algebraic numbers is countable.

Proof:- Let A = the set of all polynomials with integer coefficients, and let A_i = set of all polynomials of degree i with integer coefficients. Then $A = \bigcup_{i=1} A_i$.

Let $\mathbb{Z}^{i+1} = \{(a_0, a_1, \dots, a_i) : a_j \in \mathbb{Z}, 0 \le j \le i\}$.

Define $h_i : A_i \to \mathbb{Z}^{i+1}$ by

$h_i(a_0 z^i + a_1 z^{i-1} + \dots + a_i) = (a_0, a_1, \dots, a_i)$.

Since h_i is a 1-1 correspondence from A_i to $\mathbb{Z}^{i+1}$, and by Thm. 2.13 $\mathbb{Z}^{i+1}$ is countable, we find A_i is countable.

Since A is a countable union of countable sets, it is countable.

Now every polynomial of degree i has atmost i roots.
Let us write the elements of A_i as a sequence
$P_1, P_2, \ldots$ For any $k \in \mathbb{N}$, P_k has at most i roots.
Let r_1 = roots of P_1, r_2 = roots of P_2 and so on.
Let $R_i = \bigcup_{j=1}^{\infty} r_j$. Then R_i is countable.
For each A_i, there is an R_i.
Let $R = \bigcup_{i=1}^{\infty} R_i$ be the set of all algebraic numbers.
Then R is countable.

$\boxtimes$

2) (a) E° is always open.

<u>Proof</u>: Let $x \in E^\circ$, that is x is Interior point of E°.

<u>Claim</u> $x \in (E^\circ)^\circ$.

Now by defn., $\exists$ nbhd $N_r(x)$ with $r > 0 \ni N_r(x) \subset E$.

Let $y \neq x$ be a point of $N_r(x)$ so that $0 < d(x, y) < r$.

Now Let $h = r - d(x, y) > 0$.

Let z be anypoint s.t. $d(y, z) < h$.

Then
$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< d(x, y) + \{r - d(x, y)\} \\ &= r \end{aligned}$$

Hence $z \in N_r(x) \subset E$.

So $\exists$ nbhd $N_h(y)$ of $y \ni N_h(y) \subset E$.

$\Rightarrow y \in E^\circ$.

Thus, given $x \in E^\circ$, any point y with $d(x,y) < r$ lies in E°.

Hence $x \in (E^\circ)^\circ$, i.e. every pt. of E° is an interior pt. of E°.

$\Rightarrow E^\circ$ is open.

(b) "$\Rightarrow$" E is open. Thus every point of E is an interior point of E. $\Rightarrow E \subset E^\circ$

But $E^\circ \subset E$

$\Rightarrow E^\circ = E$.

"$\Leftarrow$" Suppose $E^\circ = E$. Thus every pt. of E is an interior pt. of E $\Rightarrow E$ is open.

(c) $G \subset E$ and G is open.

Note that $G^\circ = G$ (by part (b)).

But $G^\circ \subset E^\circ$

Hence $G \subset E^\circ$.

(d) To show $\underline{E}^{\circ c} = \overline{E^c}$

Proof: By (c), E° is the largest open set contained in E.

$$\Rightarrow E^{\circ c} = \bigcap_\alpha F_\alpha \,, \quad F_\alpha \text{ closed} \ni F_\alpha \supset E^c$$

$$= \overline{E^c}.$$

(e) No.

Consider $E = \mathbb{Q}$ in metric space $\mathbb{R}$.

Any nbhd of $x \in \mathbb{Q}$ will intersect in $\mathbb{R} \setminus \mathbb{Q}$.

$\Rightarrow$ No pt. of E is an interior pt. of E.

$\Rightarrow$ $E^\circ = \phi$

But $\bar{E} = \mathbb{R}$ & $\bar{E}^\circ = \mathbb{R}$.

(f) No.

Again take $E = \mathbb{Q}$ in m.s. $\mathbb{R}$.

$\bar{E} = \mathbb{R}$, and $\overline{E^\circ} = \bar{\phi} = \phi$.

You can also take E to be a finite set in $\mathbb{R}$.