MA 509: Tutorial 5 (2020)

1. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Show that K is compact directly from the definition.

2. We have proved in class that if $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub-collection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty. Also, we have shown that as a result of the above theorem, we get that if $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ for every $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} K_n$ is nonempty.

Show that the above results become false (in $\mathbb{R}$, for example) if the word 'compact' is replaced by 'closed' or by 'bounded'.

3. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in $\mathbb{Q}$?

4. Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?