5/10/2020

MA 509 - Tutorial 6 Solutions (2020)

① Are closures and interiors of connected sets always connected?

A set U is said to be connected if $U = A \cup B$, and $A \neq \phi$, $B \neq \phi$, then either $\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$

① The closures of connected sets <u>are</u> always connected.

Note that HW 1 prob. 2(a), $\bar{U} = \bar{A} \cup \bar{B}$.
We have $\bar{A} \neq \phi$, $\bar{B} \neq \phi$ since $A \neq \phi$, $B \neq \phi$.

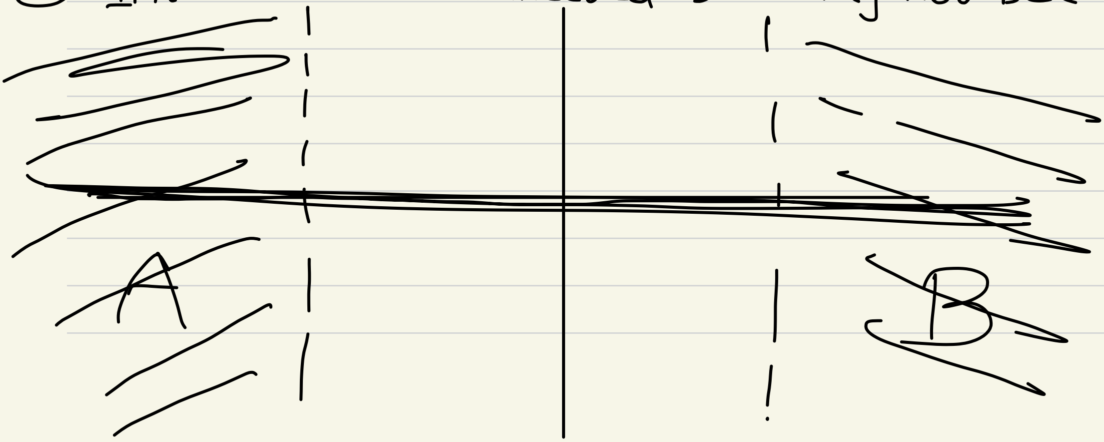
Also $\bar{A} \cap \bar{B} = \bar{A} \cap \bar{B} = \bar{A} \cap (B \cup B')$

$= (\bar{A} \cap B) \cup (\bar{A} \cap B')$ —— ①

Similarly, $\bar{A} \cap \bar{\bar{B}} = (A \cap \bar{B}) \cup (A' \cap \bar{B})$ —— ②

Since at least one of $\bar{A} \cap B$ or $A \cap \bar{B}$ is non-empty, it follows from ① & ② that $\bar{U} = \bar{A} \cup \bar{B}$ is connected.

② Interior of a connected set may not be connected



$S = A \cup B \cup \mathbb{R}$. Then $\text{int}(S) = S^\circ$ is not connected.

(2) (a) A, B closed sets of X

$A \cap B = \phi$.

Show that A & B are separated.

$A = \bar{A}$, $B = \bar{B}$

$\underline{\bar{A} \cap B} = A \cap B = \underline{\phi} = A \cap B = \underline{A \cap \bar{B}}$

(b) A, B disjoint open sets of X

$A \cap B = \phi$.

Then A and B are separated.

Assume that they are not separated.

W.l.g., say $x \in \bar{A} \cap B$

Now if $x \in A$ then $\rightarrow\leftarrow$ $x \in A \cap B = \phi$.

if $x \in A'$, x is a limit point of A ,

$\Rightarrow$ every nbhd of x intersects A in a point other than x.

Since B is open, $\exists$ nbhd U of x $\ni$ $U \subset B$.

But this nbhd U of x intersects A in a point other than x, say y.

Then $y \in A \cap B = \phi$
$\rightarrow\leftarrow$,

(c) Fix $p \in X$, $\delta > 0$.

$A = \{ q \in X : d(p,q) < \delta \}$

$B = \{ q \in X : d(p,q) > \delta \}$

Prove that A & B are separated.

follows from part (b).

(d) Every connected metric space with at least 2 points is uncountable.

Let $x_1, x_2 \in A$, where A is a connected metric space.

$\Rightarrow d(x_1, x_2) > 0$.

Consider $\{ d(x_1, x) : x \in A \}$

If it contains all distances (numbers) from 0 to $d(x_1, x_2)$, then there are uncountably many points.

Suppose $0<c<d(x_1,x_2) \ni c$ is not the distance from x_1 of any point in A.

Then consider $P=\{x\in A : d(x_1,x)<c\}$

$$Q=\{x\in A : d(x_1,x)>c\}$$

Then from part (c), P & Q are separated. Also, $A=P\cup Q$.

since this implies that A is not connected. $\rightarrow\leftarrow$