MA 509: Tutorial 8 (2020)

1. Investigate the behavior (convergence and divergence) of $\sum a_n$ if

$$a_n = \sqrt{n+1} - \sqrt{n},$$

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n},$$

$$a_n = \left(n^{1/n} - 1\right)^n,$$

$$a_n = \frac{1}{1+z^n}, \text{ for complex values of } z.$$

2. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

3. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

4. If $\{E_n\}$ is a sequence of closed and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\cap_{n=1}^{\infty} E_n$ consists of exactly one point.

5. Find the radius of convergence of each of the following series:

(a) $\sum n^3 z^n$, (b) $\sum \frac{2^n z^n}{n!}$,

(c) $\sum \frac{2^n z^n}{n^2}$, (d) $\sum \frac{n^3 z^n}{3^n}$.

6. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

(Hint: For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.