MA 623: Homework 2 (Due February 15)

(Note: Justify all the relevant steps.)

1. For positive integers m and q, the Ramanujan sum $c_q(m)$ is defined by

$$c_q(m) := \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{2\pi ikm/q}.$$

(a) Prove that

$$c_q(m) = \sum_{d|(m,q)} d\mu\left(\frac{q}{d}\right).$$

Now define the function $M(x)$ by

$$M(x) := \sum_{j \le x} \mu(j).$$

(b) Prove that

$$\sum_{q=1}^{n} c_q(m) = \sum_{d|m} dM\left(\frac{n}{d}\right),$$

and, in particular, that

$$\sum_{q=1}^{m} c_q(m) = \sum_{d|m} dM\left(\frac{m}{d}\right).$$

(c) Prove that

$$M(m) = m\sum_{d|m} \frac{\mu(m/d)}{d} \sum_{q=1}^{d} c_q(d).$$

(d) Prove that

$$\sum_{m=1}^{n} c_q(m) = \sum_{d|q} d\mu\left(\frac{q}{d}\right)\left\lfloor\frac{n}{d}\right\rfloor.$$

2. A positive integer n is called squarefull if it satisfies

$$p|n \implies p^2|n.$$

(Note that $n = 1$ is squarefull according to this definition, since 1 has no prime divisors and the above implication is therefore vacuously true.) Show that n is squarefull if and only if it can be written in the form $n = a^2b^3$ with $a, b \in \mathbb{N}$.

3. Let $d(n)$ denote the divisor function of n, that is, $d(n)$ = the number of positive divisors of n. Given an arithmetic function f such that $\sum_{n=1}^{\infty} |f(n)|d(n) < \infty$, define its "transform" $\hat{f}$ by

$$\hat{f}(j) = \sum_{n=1}^{\infty} f(nj) \quad (j \in \mathbb{N}).$$

Find (with proof) the corresponding "inverse transform", that is, a formula expressing $f(j)$ in terms of the values $\hat{f}(n)$.

4. Let φ and Λ denote the Euler totient and von Mangoldt functions respectively. Show that

$$\left(\varphi^{-1} * \Lambda\right)(n) = n \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{\log d}{d}.$$