MA 623: Homework 4 (Due: April 16)

(Note: Justify all the relevant steps.)

1. Let ϕ denote the Euler totient function. For each $j \in \mathbb{N}$, define

$$a(j) = \left|\{n \in \mathbb{N} : \phi(n) = j\}\right|.$$

Let $\sigma =$Re(s). Assuming that the series $\sum_{n=1}^{\infty} \varphi(n)^{-s}$ converges absolutely for $\sigma > 1$, prove that the identity

$$\sum_{j=1}^{\infty} \frac{a(j)}{j^s} = \sum_{n=1}^{\infty} \frac{1}{\varphi(n)^s} = \zeta(s) \prod_p \left(1 + (p-1)^{-s} - p^{-s}\right)$$

holds for $\sigma > 1$. You can also assume that the product over p on the right side also converges absolutely for $\sigma > 1$.

2. Let $a(n)$ be the greatest odd divisor of n. Prove that for $\sigma > 2$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{1 - 2^{1-s}}{1 - 2^{-s}} \zeta(s-1).$$