MA 631: Homework 1 (Due September 15)

(Note: Justify all the relevant steps.)

1. Prove the following properties satisfied by the Bernoulli numbers/Bernoulli polynomials:

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}$$

$$B_n(1-x) = (-1)^n B_n(x)$$

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n$$

$$\int_a^x B_n(t)\,dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1}.$$

2. Prove the following formulas for $n \in \mathbb{N}$:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{1-2n})B_{2n}}{2(2n)!},$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{-2n})B_{2n}}{2(2n)!}.$$

3. Evaluate $\displaystyle\int_0^{\infty} \frac{x^a}{a^x}\,dx$ for $\mathrm{Re}(\log a) > 0$ and $\mathrm{Re}(a) > -1$.

4. Obtain an estimate for the sum $\sum_{n \le x} \frac{\log n}{n}$ with error term $O\left(\frac{\log x}{x}\right)$. You do not need to "evaluate" some constants that appear in the formula, but do express them as sums or products.

5. (a) Show that $\frac{1}{\phi} = \frac{1}{N} * f$, where $f = \frac{\mu^2}{N \cdot \phi}$.
(b) Show that $\sum_{n=1}^{\infty} f(n) = O(1)$.
(c) If $x \ge 2$, show that

$$\sum_{n \le x} \frac{1}{\phi(n)} = O(\log x).$$