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MA 631 – SPECIAL FUNCTIONS – Lec. 10

We have

$$\lim_{n\to\infty}\Gamma(z,n) = \int_0^\infty \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

$$= \int_0^\infty e^{-t} t^{z-1}\,dt$$

$$= \Gamma(z) \qquad \left(\text{using } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\right)$$

$$\Rightarrow \lim_{n\to\infty}\Gamma(z,n) = \Gamma(z) \quad \text{———} \quad (a)$$

Again consider

$$\Gamma(z,n) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

Let $t = nu$, $dt = n\,du$

when $t=0$, $u=0$, and when $t=n$, $u=1$

$$\Gamma(z,n) = \int_0^1 (1-u)^n (nu)^{z-1}\, n\,du$$

$$= n^z \int_0^1 u^{z-1}(1-u)^{(n+1)-1}\,du$$

$$= n^z B(z,n+1)$$

$$= n^z \frac{\Gamma(z)\Gamma(n+1)}{\Gamma(z+n+1)}$$

$$= \frac{n^z\, n!\, \Gamma\cancel{(z)}}{(z+n)(z+n-1)\cdots z\,\Gamma\cancel{(z)}}$$

Now let $n \to \infty$ so as to get

$$\lim_{n\to\infty} \Gamma(z,n) = \lim_{n\to\infty} \frac{n^z \, n!}{z(z+1)\cdots(z+n)} \quad \text{—— (b)}$$

From (a) & (b),

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z \, n!}{z(z+1)\cdots(z+n)} .$$

<u>Logarithmic derivative of gamma function</u>

$$\psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} .$$

(psi function or digamma function)

<u>Thm. 15</u> $\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right)$,

where $z \neq 0, -1, -2, \ldots\ldots$.

Proof: We know that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$

$$-\log \Gamma(z) = \log z + \gamma z + \sum_{n=1}^{\infty} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n} \right)$$

(From what we did yesterday, the above series conv. uniformly on any cpt. subsets of the region $\mathbb{C} \setminus \{0,-1,-2,\ldots\}$.)

Hence differentiating both sides w.r.t. z, we have

$$-\psi(z) = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{(1+z/n)}\cdot\frac{1}{n} - \frac{1}{n}\right)$$

$$= \frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$

(Absolutely conv., hence we can rearrange).

$$\Rightarrow \psi(z) = -\gamma - \frac{1}{z} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{z+n+1}\right)$$

$$= -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right).$$

$\boxtimes$

Thm. 16 $\psi(z+1) = \psi(z) + \frac{1}{z}$

1st proof: $\Gamma(z+1) = z\Gamma(z)$

$$\Rightarrow \frac{d}{dz}\log\Gamma(z+1) = \frac{d}{dz}\log z + \frac{d}{dz}\log\Gamma(z)$$

$$\Rightarrow \psi(z+1) = \frac{1}{z} + \psi(z)$$

$\boxtimes$

2nd proof: $\psi(z+1) - \psi(z)$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z+1}\right) - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n+z+1} \right)$$

$$= \lim_{m \to \infty} \sum_{n=0}^{m} \left(\frac{1}{n+z} - \frac{1}{n+z+1} \right)$$

$$= \lim_{m \to \infty} \frac{1}{z} - \cancel{\frac{1}{z+1}} + \cancel{\frac{1}{z+1}} - \cancel{\frac{1}{z+2}} + \cancel{\cdots} + \cancel{\frac{1}{m+z}} - \frac{1}{m+z+1}$$

$$= \frac{1}{z}$$

Special values of $\psi(z)$:

① $\psi(1) = -\gamma$

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right)$$

$$\Rightarrow \psi(1) = -\gamma$$

② $$\psi(k+1) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{k+1+n} \right)$$

$$= -\gamma + \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right).$$

Another proof that $\Gamma(x)$ is log-convex on $(0,\infty)$

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{z+n}\right)$$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} .$$

Hence for $x > 0$,

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} > 0 .$$

$$\Rightarrow \frac{d^2}{dx^2} \log \Gamma(x) > 0$$

$\Rightarrow \log \Gamma(x)$ is convex on $(0, \infty)$. $\boxtimes$

Chapter 3

The Riemann zeta function

We have seen that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ &

more generally

$\sum_{n=1}^{\infty} \frac{1}{n^{2m}}$ is a rational multiple of π^{2m}.

Riemann zeta function is defined for $Re(s)>1$ by $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$.

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}} < \infty \quad \text{for } Re(s)>1,$$

So $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely & uniformly in $Re(s)>1$.

- Euler product for $\zeta(s)$: For $Re(s)>1$,

$$\zeta(s) = \prod_{p \text{ prime}}^{\infty} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- The Riemann zeta function can be analytically continued ~~to~~ the entire complex plane except for a simple pole at $s=1$ of residue 1.

- Riemann's functional equation for $\zeta(s)$: For all $s \in \mathbb{C}$,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Implications:

① First of all, from the Euler product of $\zeta(s)$, it can be seen that $\zeta(s) \neq 0$ for $Re(s) > 1$.

② $\zeta(1+it) \neq 0$, for $t \neq 0$ (used in the proof of prime number theorem)

③ The functional equation tells us that $\zeta(s) = 0$ for $s = -2, -4, -6, \ldots$
These are called the trivial zeros of $\zeta(s)$.