31/1/22

MA 631 – SPECIAL FUNCTIONS – Lec. 12

Chapter 4 – Hypergeometric functions

Pochhammer symbol (or shifted factorial)

- $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$
$$= \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geq 0,$$

- $(a)_0 = 1$,

- $(1)_n = (1)(2)(3)\cdots(1+n-1)$
$$= n!$$

- $$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$$

- $$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!}, \quad |z|<1$$

- $${}_2F_1\left(\begin{matrix} a \;\; b \\ c \end{matrix}; z\right) \quad \left(\text{or } {}_2F_1(a,b;c;z)\right)$$
$$:= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n\, n!}$$

Gauss Hypergeometric series

$$= 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2} + \cdots$$

Radius of convergence of ${}_2F_1(a,b;c;z)$:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(a)_n(b)_n}{(c)_n\, n!}}{\dfrac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(c+n)(n+1)}{(a+n)(b+n)}\right| = \lim_{n\to\infty}\left|\frac{(1+c/n)(1+1/n)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}\right|$$

$$= 1$$

$\Rightarrow R = 1.$

- ${}_2F_1(a,b;c;z)$ converges absolutely for $|z|<1$ and diverges for $|z|>1$.

- It is a delicate question to know what happens on $|z|=1$.

- Since ${}_2F_1(a,b;c;z)$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b+n)}{\Gamma(b)}\cdot\frac{\Gamma(c)}{\Gamma(c+n)}\frac{z^n}{n!},$$

we see that this definition is valid for $a,b\in\mathbb{C}$, $c\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$ and $|z|<1$.

But $\frac{1}{\Gamma(c)}\, {}_2F_1(a,b;c;z)$ is an entire function of a,b as well as c,

[Regularized Hypergeometric function]

- $$\lim_{c\to 0} \frac{1}{\Gamma(c)}\, {}_2F_1(a,b;c;z) = abz \cdot {}_2F_1(a+1, b+1; 2; z)$$

<u>Proof:</u> $$\lim_{c\to 0} \frac{1}{\Gamma(c)}\, {}_2F_1(a,b;c;z)$$

$$= \lim_{c\to 0} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$= \lim_{c\to 0} \left\{ \frac{1}{\Gamma(c)} + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n)} \frac{z^n}{n!} \right\}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(n-1)!\, n!}$$ { Unif. conv. inside the disk of conv. }.

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1} z^{n+1}}{n!\, (n+1)!}$$

$$= abz \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n z^n}{(2)_n\, n!}$$

$$\left((2)_n = 2\cdot 3 \cdot \cdots (2+n-1) = (n+1)!\right)$$

$$= abz\, {}_2F_1(a+1, b+1; 2; z)$$

• When a or b are non-positive integers, then the hypergeometric series terminates, and ${}_2F_1$ reduces to a polynomial in z.

$$ {}_2F_1\left(\begin{matrix}-m, & b \\ c & \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(-m)_n (b)_n z^n}{(c)_n \, n!} $$

Note that $(-m)_n = \begin{cases} (-m)(-m+1)\cdots(-m+n-1), & n \le m \\ 0, & n > m. \end{cases}$

Also, $(-m)(-m+1)\cdots\cdots(-m+n-1)$

$$ = (-1)^n m(m-1)\cdots(m-(n-1)) $$

$$ = (-1)^n \frac{m!}{(m-n)!} $$

$$ = (-1)^n \binom{m}{n} n! $$

Thus ${}_2F_1\left(\begin{matrix}-m, & b \\ c & \end{matrix}; z\right)$ is a poly. of deg. $\le m$.

Special cases of ${}_2F_1(a, b; c; z)$:

1. ${}_2F_1(a, b; b; z) = (1-z)^{-a}$
2. ${}_2F_1(1, 1; 2; z) = \dfrac{-\ln(1-z)}{z}$.

$$③\quad {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; z^2\right) = \underbrace{\frac{1}{2z}\ln\left(\frac{1+z}{1-z}\right)}_{=\tanh^{-1}(z)}$$

$$④\quad {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2\right) = \frac{\tan^{-1}(z)}{z}$$

$$⑤\quad {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \frac{\sin^{-1}(z)}{z}$$

$$⑥\quad {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; -z^2\right) = \frac{\ln\left(z+\sqrt{1+z^2}\right)}{z}$$

Euler's integral representation for ${}_2F_1(a, b; c; z)$

Thm. 4.1 For $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ & $|\arg(1-z)| < \pi$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-1}(1-tz)^{-a}\,dt$$