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MA 631 – SPECIAL FUNCTIONS – Lec. 14

Tutorial 3 – Problem 2

If $Re(m) > 0$, $Re(n) > 0$, then

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(b+cx)^{m+n}} = \frac{1}{b^n (b+c)^m} B(m,n)$$

[P. R. Taylor, A functional equation for Epstein zeta function, Q. J. Math. 1940]

Proof: $\frac{1}{c^{m+n}} \int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(a+x)^{m+n}}$,

where $a = b/c$.

Change of variable: $\frac{x}{a+x} = \frac{t}{a+1}$

when $x=0$, $t=0$, and when $x=1$, $t=1$.

$$\Rightarrow \frac{a+x}{x} = \frac{a+1}{t}$$

$$\Rightarrow \frac{a+x-x}{x} = \frac{a+1-t}{t}$$

$$\Rightarrow \frac{a}{x} = \frac{a+1-t}{t}$$

$$\Rightarrow x = \frac{at}{a+1-t} \Rightarrow dx = \frac{\left[(a+1-t)a - at(-1)\right]dt}{(a+1-t)^2}$$

$$\Rightarrow dx = \frac{a(a+1)}{(a+1-t)^2}\,dt.$$

$$1-x = 1-\frac{at}{a+1-t} = \frac{a+1-t-at}{a+1-t}$$

$$= \frac{(a+1)(1-t)}{a+1-t}$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}\,dx$$

$$= \frac{1}{c^{m+n}} \int_0^1 \frac{\left(\frac{at}{a+1-t}\right)^{m-1}\left(\frac{(a+1)(1-t)}{a+1-t}\right)^{n-1}}{\left(a+\frac{at}{a+1-t}\right)^{m+n}} \frac{a(a+1)\,dt}{(a+1-t)^2}$$

$$= \frac{1}{c^{m+n}}\, a^{m} \frac{(a+1)^{n}}{(a+1)^{m+n}} \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= \frac{1}{c^{m+n}(a+1)^{m}a^{n}} B(m,n)$$

$$= \frac{1}{b^{n}c^{m}\left(\frac{b}{c}+1\right)^{m}} B(m,n)$$

$$= \frac{1}{b^{n}(b+c)^{m}} B(m,n).$$

Symmetry in ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$

For $|z|<1$, ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} b, a \\ c \end{matrix}; z\right)$.

For $|\arg(1-z)|<\pi$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt. $$

If $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, and $|\arg(1-z)|<\pi$,

$$ {}_2F_1\left(\begin{matrix} b, a \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\,dt $$

Erdelyi (1937) — Q.J.Math, (p.267–277)

For $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)|<\pi$

${}_2F_1(a, b; c; z)$

$$ = \frac{\Gamma^2(c)}{\Gamma(b)\Gamma(c-b)\Gamma(a)\Gamma(c-a)} \qquad \longrightarrow (*) $$

$$ \times \int_0^1\int_0^1 t^{b-1}(1-t)^{c-b-1} y^{a-1}(1-y)^{c-a-1}(1-tyz)^{-c}\,dy\,dt $$

$$ \Rightarrow {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z). $$

(*) Identity theorem will also establish the above equality outside of the unit disc.

Proof of (*)

Let $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$, $|\arg(1-z)| < \pi$, then

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \quad \text{— (1)} $$

Now if $\operatorname{Re}(c) > 0$, $\operatorname{Re}(c-a) > 0$,

$$ \int_0^1 \frac{y^{a-1}(1-y)^{c-a-1}}{(1-zty)^{c}}\, dy = \frac{1}{(1-zt)^{a}} B(a, c-a) \quad \text{— (2)} $$

From (1) & (2),

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} $$

$$ \times \int_0^1 t^{b-1}(1-t)^{c-b-1} \int_0^1 y^{a-1}(1-y)^{c-a-1}(1-ytz)^{-c}\, dy\, dt $$

Gauss' representation for ${}_2F_1(a, b; c; 1)$

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, $$

whenever $\operatorname{Re}(c-a-b) > 0$, $\operatorname{Re}(c) > \operatorname{Re}(a) > 0$
$\operatorname{Re}(c) > \operatorname{Re}(b) > 0$.

Absolute Convergence of ${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; 1\right)$ whenever $\mathrm{Re}(c-a-b)>0$:

Let $\delta := \frac{1}{2}\mathrm{Re}(c-a-b) > 0$.

Compare the terms of the series $1+\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!}$ with corresponding terms of $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$.

To that end,

$$L = \lim_{n\to\infty} \left| \frac{n^{1+\delta}(a)_n (b)_n}{(c)_n \quad n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(a)_n}{(n-1)^a (n-1)!} \quad \frac{(b)_n}{(n-1)^b (n-1)!} \quad \frac{(n-1)!(n-1)^c}{(c)_n} \cdot n^{\delta} (n-1)^{a+b-c} \right|.$$

Note that

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n^z n!}{(z)_{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n-1)^z (n-1)!}{(z)_n}$$

Then $L = \left| \frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} \Gamma(c) \right| \cdot \lim_{n \to \infty} n^{\delta} (n-1)^{a+b-c}$

$$= \left| \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \right| \cdot \lim_{n\to\infty} n^{\delta + Re(a+b-c)} \left| \left(1 - \frac{1}{n}\right) \right|^{Re(a+b-c)} .$$

$$= \left| \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \right| \lim_{n\to\infty} \frac{1}{n^{\underbrace{Re(c-a-b)-\delta}}}$$

$$= 0 < 1 \qquad \left(\because Re(c-a-b) - \delta = \frac{1}{2} Re(c-a-b) > 0 \right)$$

By limit comparison test

$1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n}$ converges if $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$ converges.

But $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$ converges.

Hence ${}_2F_1(a, b; c; 1)$ converges when $Re(c-a-b) > 0$.