8/11/22

MA 631 – SPECIAL FUNCTIONS – Lec. 16

We know that ${}_2F_1(a,b;c;z)$ satisfies the DE

$$\theta(\theta+c-1)\,{}_2F_1(a,b;c;z)$$
$$= z(\theta+a)(\theta+b)\,{}_2F_1(a,b;c;z). \longrightarrow (*)$$

Traditional form: $\qquad\qquad (F := {}_2F_1)$

$$\begin{aligned}
\text{LHS} &= \theta(\theta+c-1)F \\
&= \theta(\theta F + (c-1)F) \\
&= \theta(zF' + (c-1)F) \\
&= \theta(zF') + (c-1)\theta F \\
&= z(zF'' + F') + (c-1)zF' \\
&= z^2F'' + czF'.
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= z(\theta+a)(\theta+b)F \\
&= z(\theta+a)(zF' + bF) \\
&= z[\theta(zF'+bF) + a(zF'+bF)] \\
&= z[z(zF''+F'+bF') + azF' + abF] \\
&= z[z^2F'' + (a+b+1)zF' + abF]
\end{aligned}$$

$$\Rightarrow z(zF'' + cF') = z(z^2F'' + (a+b+1)zF' + abF)$$

$$\Rightarrow z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0.$$

Thus ${}_2F_1(a,b;c;z)$ is a solution of the differential equation $(**)$

$$\boxed{z(1-z)\frac{d^2y}{dz^2} + (c-(a+b+1)z)\frac{dy}{dz} - aby = 0}$$

Hypergeometric differential eqn (HDE)

We now show that $z^{1-c}G$, where G is again a hypergeometric function, is also a sol^n of the HDE.

Let $H = z^{1-c}G$. Then

$$\theta(\theta+c-1)H$$

$$= \theta(\theta+c-1)(z^{1-c}G)$$

$$= \theta\left[\theta(z^{1-c}G) + (c-1)z^{1-c}G\right]$$

$$= \theta\left[z\left(z^{1-c}G' + (1-c)z^{-c}G\right) + (c-1)z^{1-c}G\right]$$

$$= \theta\left[z^{2-c}G'\right]$$

$$= z\left((2-c)z^{1-c}G' + z^{2-c}G''\right)$$

$$= z^{1-c}\left(z^2G'' + z(2-c)G'\right) \quad \text{—— } (1)$$

Now consider

$$\theta(\theta+1-c)G$$

$$= \theta(zG' + (1-c)G)$$

$$= z(zG'' + G' + (1-c)G')$$

$$= z^2G'' + (2-c)zG'$$

$$= (\theta+1-c)\theta G \quad \text{—— } (2)$$

From (1) & (2),

$$\theta(\theta+c-1)(z^{1-c}G) = z^{1-c}(\theta+1-c)\theta(G) \quad \text{—— } (3)$$

Also, $z(\theta+a)(\theta+b)(z^{1-c}G)$
$= z^{2-c}(\theta+a+1-c)(\theta+b+1-c)G$ —— 4

From *, 3 & 4,

$\theta(\theta+1-c)G = z(\theta+a-c+1)(\theta+b-c+1)G$

$\Rightarrow \theta(\theta+(2-c)-1)G = z(\theta+a-c+1)(\theta+b-c+1)G$ —— 5

From $\theta(\theta+c-1)\,{}_2F_1(a,b;c;z)$
$= z(\theta+a)(\theta+b)\,{}_2F_1(a,b;c;z)$

5 is a reparametrization of HDE in * of which ${}_2F_1\left(\begin{matrix}a-c+1,\ b-c+1\\ 2-c\end{matrix};z\right)$ is a solution.

Thus, in addition to ${}_2F_1(a,b;c;z)$ the second solution of * or ** is $z^{1-c}\,{}_2F_1\left(\begin{matrix}a-c+1,\ b-c+1\\ 2-c\end{matrix};z\right)$,

The general solution is
$P\cdot{}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};z\right) + Q\, z^{1-c}\,{}_2F_1\left(\begin{matrix}a-c+1,\ b-c+1\\ 2-c\end{matrix};z\right)$,
where P and Q are independent of z. —— 6

Applications:

1. With the help of HDE, we can express a hypergeometric function with arguments $1-z$ or $1/z$ in terms of functions with argument z.

Let us introduce a new variable $z' = 1 - z$ in the HDE $z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0$.

Since $z = 1 - z'$,

$$\frac{dF}{dz'} = \frac{dF}{dz} \cdot \frac{dz}{dz'} = -F'.$$

Also
$$\frac{d^2F}{dz'^2} = \frac{d}{dz'}\left(\frac{dF}{dz'}\right) = \frac{d}{dz'}(-F')$$

$$= \frac{d}{dz}(-F') \cdot \frac{dz}{dz'} = F''.$$

$\Rightarrow$ The given HDE becomes

$$z'(1-z')F'' - (c - (a+b+1)(1-z'))F' - abF = 0.$$

$$\Downarrow$$

$$z'(1-z')F'' + (a+b+1-c-(a+b+1)z')F' - abF = 0$$

Hence a solution is ${}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix}; 1-z\right)$ ——— (7)

From (6) & (7),

$$\boxed{\begin{gathered} {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix}; 1-z\right) \\ = P \cdot {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right) + Q\, z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1, \ b-c+1 \\ 2-c \end{matrix}; z\right). \\ \text{———} \ (8) \end{gathered}}$$

$\mathrm{Re}(\cancel{2} - \cancel{c} - a + \cancel{c} - \cancel{1} - b + c - \cancel{1}) > 0$

$\mathrm{Re}(c - a - b)$

Finding P & Q :

Let $\mathrm{Re}(c) < 1$ & $\mathrm{Re}(c-a-b) > 0$.
Let $z = 0$ in (8). Then

$$\underbrace{{}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix} ; 1\right)}^{\text{converges since } \mathrm{Re}(c)<1} = P\, {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix} ; 0\right) + Q(0)$$

$$\Rightarrow P = {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix} ; 1\right)$$

$$= \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(b-c+1)\,\Gamma(a-c+1)}.$$

[Gauss' formula
$${}_2F_1\left(\begin{matrix} A,\ B \\ C \end{matrix} ; 1\right) = \frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)}$$
for $\mathrm{Re}(C-A-B) > 0$]

Tutorial problem:
Prove that

$$Q = \frac{\Gamma(c-1)\,\Gamma(a+b+1-c)}{\Gamma(a)\,\Gamma(b)}.$$