9/11/22

MA 631 – SPECIAL FUNCTIONS – Lec. 17

$$
{}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix}; 1-z\right)
$$

$$
= P \cdot {}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right) + Q\, z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1, \; b-c+1 \\ 2-c \end{matrix}; z\right),
$$

where

$$
P = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(b-c+1)\,\Gamma(a-c+1)}, \quad Q = \frac{\Gamma(c-1)\,\Gamma(a+b+1-c)}{\Gamma(a)\,\Gamma(b)}
$$

for $Re(c-a-b) > 0$ & $Re(c) < 1$.

By analytic continuation in parameters a, b & c, we can now relax the above restrictions.

In the above formula, replace z by $1-z$ and c by $a+b+1-c$. This results in

Thm. 4.4

$$
{}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}; z\right) \qquad \text{—} \; \text{I}
$$

$$
= A\, {}_2F_1\left(\begin{matrix} a \quad b \\ a+b+1-c \end{matrix}; 1-z\right) + B\, z^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-b, \; c-a \\ 1+c-a-b \end{matrix}; 1-z\right),
$$

where

$$
A = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \quad B = \frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \; \& \; |\arg(1-z)| < \pi.
$$

Application: When $\frac{1}{2} < z < 1$, then the series defn. of ${}_2F_1(a,b;c;z)$ is not convergent fast enough as far as numerics is concerned.

In this case, I is helpful since $0 < 1-z < \frac{1}{2}$.

Cor. 4.5 (Pfaff) Let $n \in \mathbb{Z}^+$. Then

$$ {}_2F_1\left(\begin{matrix}-n, & b \\ & c\end{matrix}; z\right) = \frac{(c-b)_n}{(c)_n} \cdot {}_2F_1\left(\begin{matrix}-n, & b \\ & b+1-n-c\end{matrix}; 1-z\right) $$

Proof: Let $a = -n$ in Thm. 4.4.
Observe that $\dfrac{1}{\Gamma(-n)} = 0$, hence $B = 0$.

Therefore,

$$ {}_2F_1\left(\begin{matrix}-n, & b \\ & c\end{matrix}; z\right) = \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-b)} \, {}_2F_1\left(\begin{matrix}-n & b \\ & -n+b+1-c\end{matrix}; 1-z\right) $$

$$ = \frac{(c-b)_n}{(c)_n} \, {}_2F_1\left(\begin{matrix}-n, & b \\ & b+1-n-c\end{matrix}; 1-z\right). $$

Cor. (4.6) (<u>Chu - Vandermonde</u>) Let $n \in \mathbb{Z}^+$.

$$ {}_2F_1\left(\begin{matrix}-n, & b \\ & c\end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n} $$

1^{st} Proof: Put $z = 1$ in Cor. 4.4.

2^{nd} proof: By Gauss' formula,

$$ {}_2F_1\left(\begin{matrix}-n, & b \\ & c\end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-(-n)-b)}{\Gamma(c-(-n))\,\Gamma(c-b)} $$

$$ = \frac{\Gamma(c)\Gamma(c-b+n)}{\Gamma(c+n)\Gamma(c-b)} = \frac{(c-b)_n}{(c)_n}, $$

Thm, 4.6 For, $m, n, k \in \mathbb{Z}^+$,

$$\sum_{h=0}^{k} \binom{m}{h} \binom{n}{k-h} = \binom{m+n}{k}$$

Proof: Both the sides are counting the number of ways of choosing k objects out of m+n objects. On the left, this is done by considering 2 groups of the m+n objects one consisting m & other of n. Then we choose h objects out of m objects and hence we have k-h objects left to be chosen out of n ones where $0 \leq h \leq k$. This establishes the formula.

Some more transformations

$$_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = C \cdot (-z)^{-a} \, _2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix}; \frac{1}{z}\right)$$

$$+ D(-z)^{-b} \, _2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{z}\right),$$

where $C = \dfrac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}$ & $D = \dfrac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}$,

valid for $|\arg(-z)| < \pi$.

Application: It allows us to use the series representation in $|z| > 1$. This is use for calculating asymptotic behavior of $_2F_1(a, b; c; z)$ as $z \to \infty$.

Remark: For $|1-z|<1$, $|\arg(1-z)|<\pi$,

$$ {}_2F_1\left(\begin{matrix} a \quad b \\ a+b \end{matrix}; z\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} C_n (1-z)^n, $$

where

$$ C_n = 2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-z). $$

Asymptotic expansion

Def. Let F be a function of a real or complex variable z. Let $\sum_{k=0}^{\infty} a_k z^{-k}$ denote a (convergent or divergent) formal power series of which the first n terms we denote by $S_n(z)$. Let $R_n(z) = F(z) - S_n(z)$, that is,

$$ F(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots + \frac{a_{n-1}}{z^{n-1}} + R_n(z), $$

$n = 0, 1, 2, 3, \ldots$. (When $n=0$, $F(z) = R_0(z)$)
Assume that for each $n = 0, 1, 2, \ldots$, the following relation holds:

$$ R_n(z) = O(z^{-n}) \quad \text{as } z \to \infty $$

in some unbounded domain Δ. Then $\sum_{k=0}^{\infty} a_k z^{-k}$ is called an asymptotic expansion of $F(z)$ & is denoted by

$$ F(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \quad z \to \infty, \quad z \in \Delta. $$

Problem: Find asymptotic expansion of

$$S(x) := \int_0^\infty \frac{dt}{(1+t)^{1/3}(x+t)} \quad \text{as } x \to \infty.$$

$$= \left[\frac{1}{x+t}(1+t)^{2/3} \cdot \frac{3}{2}\right]_0^\infty + \frac{3}{2}\int_0^\infty \frac{1}{(x+t)^2}(1+t)^{2/3}\,dt$$

$$= -\frac{3}{2x} + \frac{3}{2}\left\{\left[\frac{1}{(x+t)^2}(1+t)^{5/3} \cdot \frac{3}{5}\right]_0^\infty + 2\times\frac{3}{5}\int_0^\infty \frac{(1+t)^{5/3}}{(x+t)^3}\,dt\right\}$$

$$= \frac{-3}{2x} - \frac{3^2}{2\times 5\, x^2} + \frac{3^2}{2\times 5}\int_0^\infty \frac{(1+t)^{5/3}}{(x+t)^3}\,dt.$$

Iterating this process, we see that

$$S(x) \sim -\sum_{n=1}^{\infty} \frac{3^n (n-1)!}{2 \cdot 5 \cdots (3n-1)\, x^n} \quad \text{as } x \to \infty.$$