14/1/22

MA 631 – SPECIAL FUNCTIONS – Lec. 18

Problem: Find asymptotic expansion of

$$S(x) := \int_0^\infty \frac{dt}{(1+t)^{1/3}(x+t)} \quad \text{as } x \to \infty.$$

$$= \left[\frac{1}{x+t}(1+t)^{2/3}\cdot\frac{3}{2}\right]_0^\infty + \frac{3}{2}\int_0^\infty \frac{1}{(x+t)^2}(1+t)^{2/3}\,dt$$

$$= -\frac{3}{2x} + \frac{3}{2}\left\{\left[\frac{1}{(x+t)^2}(1+t)^{5/3}\cdot\frac{3}{5}\right]_0^\infty + 2\times\frac{3}{5}\int_0^\infty \frac{(1+t)^{5/3}}{(x+t)^3}\,dt\right\}$$

$$= -\frac{3}{2x} - \frac{3^2}{2\times 5\, x^2} + \frac{3^2}{2\times 5}\int_0^\infty \frac{(1+t)^{5/3}}{(x+t)^3}\,dt.$$

Iterating this process, we see that

$$S(x) \sim -\sum_{n=1}^{\infty} \frac{3^n (n-1)!}{2\cdot 5\cdots(3n-1)\,x^n} \quad \text{as } x\to\infty.$$

$$= -\frac{3}{2x}\,{}_2F_1\left(1,1,\frac{5}{3};\frac{1}{x}\right)$$

This is WRONG! This is because, the integral is a positive real number where as the expression $-\frac{3}{2x}\,{}_2F_1\left(1,1,\frac{5}{3};\frac{1}{x}\right)$ is clearly negative.

WHAT WENT WRONG?

Note that integration by parts gives

$$S(x) = -\sum_{n=1}^{N-1} \frac{3^n (n-1)!}{2\cdot 5 \cdots (3n-1)} \cdot \frac{1}{x^n} + R_N(x),$$

where $$R_N(x) = \frac{3^{N-1}(N-1)!}{2\cdot 5\cdot 8 \cdots (3N-4)} \cdot \int_0^\infty \frac{(1+t)^{N-4/3}}{(x+t)^N}\, dt.$$

<u>Claim</u> $R_N(x) = O(x^{-N})$ is not true, as $x \to \infty$.

Proof of the claim:

$t \le 1+t \le x+t$ for $x \geqslant 1$

$$\underbrace{\int_0^\infty \frac{t^{N-4/3}\, dt}{(x+t)^N}}_{(*)} \le \int_0^\infty \frac{(1+t)^{N-4/3}}{(x+t)^N}\, dt \le \int_0^\infty \frac{(x+t)^{N-\frac{4}{3}}}{(x+t)^N}\, dt$$

$$= \int_0^\infty (x+t)^{-4/3}\, dt$$

$$= -3\left[(x+t)^{-1/3}\right]_0^\infty$$

$$= \frac{3}{x^{1/3}}$$

Let $t = xu$ in (*). Then

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^\infty \frac{t^{N-4/3}}{(x+t)^N}\,dt = \frac{x^{N-1/3}}{x^N}\int_0^\infty \frac{u^{N-4/3}\,du}{(1+u)^N}$$

$$= x^{-1/3}\int_0^\infty \frac{u^{(N-1/3)-1}\,du}{(1+u)^{(N-\frac{1}{3})+\frac{1}{3}}}$$

$$= x^{-1/3}\,\frac{\Gamma(N-\frac{1}{3})\,\Gamma(\frac{1}{3})}{\Gamma(N)}$$

$$\Rightarrow \frac{\Gamma(N-1/3)\,\Gamma(1/3)}{\Gamma(N)}\cdot\frac{1}{x^{1/3}} \leq \int_0^\infty \frac{(1+t)^{N-4/3}\,dt}{(x+t)^N} \leq \frac{3}{x^{1/3}}$$

Hence $R_N(x) = O\left(\frac{1}{x^{1/3}}\right)$ as $x\to\infty$

This violates the definition of asymptotic expansion as it should have satisfied
$R_N(x) = O\left(\frac{1}{x^{N-1}}\right)$.

Another approach:

$$S(x) := \int_0^\infty \frac{dt}{(1+t)^{1/3}(x+t)}$$

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!}, \quad |z|<1$$

Now $(1+t)^{-1/3} = t^{-1/3}\left(1+\frac{1}{t}\right)^{-1/3}$

$$= t^{-1/3} \sum_{n=0}^{\infty} \frac{(1/3)_n (-1/t)^n}{n!}$$

Hence

$$S(x) = \int_0^{\infty} \frac{t^{-1/3}}{x+t} \sum_{n=0}^{\infty} \frac{(1/3)_n (-1/t)^n}{n!}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(1/3)_n (-1)^n}{n!} \int_0^{\infty} \frac{t^{-n-1/3}}{(x+t)}\, dt$$

Let $t = xu$

$$= \sum_{n=0}^{\infty} \frac{(1/3)_n (-1)^n}{n!} \; x^{-n-1/3} \int_0^{\infty} \frac{u^{-n-1/3}}{1+u}\, du$$

$$= \sum_{n=0}^{\infty} \frac{(1/3)_n (-1)^n}{n!} \; x^{-n-1/3} \; B\left(-n+\frac{2}{3}, n+1/3\right)$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n (-1)^n}{n!} \; x^{-n-1/3} \; \Gamma\left(-n+\frac{2}{3}\right)\Gamma\left(n+1/3\right)$$

$$= \sum_{n=0}^{\infty} \frac{(1/3)_n (-1)^n x^{-n-1/3}}{n!} \; \frac{\pi}{\underbrace{\sin(\pi(n+1/3))}_{\substack{=\sin \pi n \cos \pi/3 + \cos \pi n \sin \pi/3 \\ = \frac{\sqrt{3}}{2}(-1)^n}}}$$

$$= \frac{2\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(1/3)_n \, x^{-n-1/3}}{n!}$$

$$= \frac{2\pi}{\sqrt{3}\, x^{1/3}} \, {}_2F_1\left(\begin{matrix} \frac{1}{3}, & 1 \\ & 1 \end{matrix} ; \frac{1}{x} \right).$$

This attempt is also incorrect because the integrals $\int_0^\infty \frac{t^{-n-1/3}}{x+t}\,dt$ diverge for $n \geq 1$.

Correct approach:

Consider

$$I := \int_0^\infty t^{b-1} (1+t)^{c-b-1} (1+xt)^{-a}\,dt$$

for $Re(b) > 0$ & $Re(a+1-c) > 0$.

Let $t = \frac{u}{1-u} \Rightarrow \frac{1-u}{u} = \frac{1}{t} \Longrightarrow \frac{1}{u} = \frac{1+t}{t}$

$\Rightarrow u = \frac{t}{1+t}$

$$\Rightarrow \frac{u\cancel{(-1)} - (1-\cancel{u})}{u^2}\,du = \frac{-1}{t^2}\,dt$$

$$\Rightarrow \frac{-1}{u^2}\,du = \frac{-1}{t^2}\,dt.$$

Then

$$I = \int_0^1 \left(\frac{u}{1-u}\right)^{b-1} \left(1+\frac{u}{1-u}\right)^{c-b-1} \left(1+\frac{xu}{1-u}\right)^{-a} \frac{\left(\frac{u}{1-u}\right)^2}{u^2}\,du$$

$$1-u+xu$$
$$= 1-u(1-x)$$

$$= \int_0^1 u^{b-1}(1-u)^{\cancel{-b}+\cancel{1}-c+\cancel{b}+\cancel{1}+a-\cancel{2}} \left(1-(1-x)u\right)^{-a} du$$

Note that for $Re(c) > Re(b) > 0$ & $|\arg(1-z)| < \pi$

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} ; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

Hence $c-b-1 = -c+a$

$$\Rightarrow c = a-c+b+1$$

$$= \frac{\Gamma(b)\Gamma(a-c+1)}{\Gamma(a-c+b+1)}\ {}_2F_1\left(\begin{matrix} a, & b \\ a-c+b+1 \end{matrix} ; (1-x)\right)$$

for $Re(a-c+b+1) > Re(b) > 0$ and
$|\arg(1-(1-x))| < \pi$