MA 631 – Special Functions – Lec. 1

Bernoulli numbers

Define $S_n(m) := \sum_{i=0}^{n-1} i^m$.

- $S_n(1) = \sum_{i=0}^{n-1} i^1 = 1+2+\cdots+n-1 = \frac{n(n-1)}{2}$
  $= \frac{n^2}{2} - \frac{n}{2}$
- $S_n(2) = \frac{(n-1)\,n\,(2n-1)}{6} = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$
- $S_n(3) = \left(\frac{n(n-1)}{2}\right)^2 = \frac{n^4}{4} - \frac{n^3}{2} + \frac{n^2}{4}$

Johann Bernoulli (1713)

- $S_n(m) = 1^m + 2^m + \cdots + (n-1)^m$

$$= \frac{1}{m+1}\left(B_{n+1}(m) - B_{n+1}\right)$$

$$= \frac{1}{(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} B_k \, n^{m+1-k}$$

Applications of Bernoulli numbers

- combinatorial theory
- finite difference calculus
- probability theory
- analytic number theory
- numerical analysis

- Generating function

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n,$$

where f_n is independent of t.

- Bernoulli numbers B_n are defined as follows:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

- $$\frac{z}{e^z-1} = \frac{z}{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - 1}$$

$$= \frac{\cancel{z}}{\cancel{z}\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}$$

$$= 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \cdots$$

Hence the first few Bernoulli numbers are given by

$B_0 = 1$, $B_1 = \frac{-1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$,

$B_4 = \frac{-1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, ...

Lemma 1

Let $f(z) = \dfrac{z}{e^z - 1} - 1 + \dfrac{z}{2}$;

Prove that f is an even function of z.

Proof:

1^{st} method: Show $f(-z) = f(z)$ directly.

2^{nd} method:

$$\boxed{\pi \cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}, \quad a \notin \mathbb{Z}.}$$

$$\Rightarrow \frac{\pi \cosh(\pi a)}{i \sinh(\pi a)} = \frac{1}{ia} - \sum_{n=1}^{\infty} \frac{2ia}{a^2 + n^2}$$

$$\Rightarrow \frac{\pi \left(\dfrac{e^{\pi a} + e^{-\pi a}}{2} \right)}{\left(\dfrac{e^{\pi a} - e^{-\pi a}}{2} \right)} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2}$$

$$\Rightarrow \pi \frac{e^{2\pi a} + 1}{e^{2\pi a} - 1} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2}$$

$$\Rightarrow \pi \frac{(e^{2\pi a} - 1 + 2)}{e^{2\pi a} - 1} = \text{———} '' \text{———}$$

$$= \pi + \frac{2\pi}{e^{2\pi a}-1} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2+n^2}$$

Let $2\pi a = z$. Then

$$\Rightarrow \pi + \frac{2\pi}{e^{z}-1} = \frac{2\pi}{z} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{z}{\frac{z^2}{4\pi^2}+n^2}$$

$$\Rightarrow \pi + \frac{2\pi}{e^{z}-1} = \frac{2\pi}{z} + 4\pi\sum_{n=1}^{\infty} \frac{z}{z^2+4\pi^2 n^2}$$

$$\Rightarrow \frac{1}{e^{z}-1} - \frac{1}{z} + \frac{1}{2} = 2\sum_{n=1}^{\infty} \frac{z}{z^2+4\pi^2 n^2}$$

$$\Rightarrow f(z) = 2\sum_{n=1}^{\infty} \frac{z^2}{z^2+4\pi^2 n^2}$$

$$\Rightarrow f(-z) = f(z).$$

Thm. 2 $B_{2n+1} = 0$ for every $n \in \mathbb{N}$.

From lemma 1,

$$\frac{z}{e^{z}-1} - 1 + \frac{z}{2} = \frac{-z}{e^{-z}-1} - 1 - \frac{z}{2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{B_n z^n}{n!} - 1 + \frac{z}{2} = \sum_{n=0}^{\infty} \frac{B_n (-z)^n}{n!} - 1 - \frac{z}{2}$$

$$\Rightarrow \cancel{B_0} + B_1 z + \sum_{n=2}^{\infty} \frac{B_n z^n}{n!} \cancel{-1} + \frac{z}{2}$$

$$= \cancel{B_0} - B_1 z + \sum_{n=2}^{\infty} \frac{B_n (-1)^n z^n}{n!} \cancel{-1} - \frac{z}{2}$$

$$\Rightarrow z(2B_1 + 1) + \sum_{n=2}^{\infty} \frac{B_n z^n}{n!}\left(1 - (-1)^n\right) = 0.$$

$$\Rightarrow 2B_1 + 1 = 0$$

& $B_n = 0$ for every odd $n \geq 3$.

$$\Rightarrow B_1 = \frac{-1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad \forall n \in \mathbb{N}.$$

$\blacksquare$

<u>Bernoulli polynomials</u>:

$$\frac{z e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!}, \quad |z| < 2\pi$$