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MA 631 – Special Functions – Lec. 2

Bernoulli polynomials:

$$\frac{z e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!}, \quad |z| < 2\pi$$

- when $x = 0$, $B_n(0) = B_n$.

- $B_0(x) = 1$
  $B_1(x) = x - \frac{1}{2}$
  $B_2(x) = x^2 - x + \frac{1}{6}$

& so on.

Thm. 3
(i) $\int_0^1 B_n(x)\,dx = 0 \quad \forall n \in \mathbb{N}.$

Proof: Note that

$$\frac{z e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!}$$

Integrate both sides w.r.t. x from 0 to 1:

$$\int_0^1 \frac{z e^{xz}}{e^z - 1}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!}\,dx$$

(Note that $\sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!}$ conv. unif. on compact subsets of $(0,1)$.)

$$\Rightarrow \frac{z}{e^z-1}\int_0^1 e^{xz}\,dx = \sum_{n=0}^{\infty} \frac{z^n}{n!}\int_0^1 B_n(x)\,dx$$

$$\Rightarrow \frac{z}{e^z-1}\cdot \left.\frac{e^{xz}}{z}\right|_0^1 = \sum_{n=0}^{\infty} \int_0^1 B_n(x)\,dx \, \frac{z^n}{n!}$$

$$\Rightarrow \frac{z}{e^z-1}\cdot \frac{e^z-1}{z} = \sum_{n=0}^{\infty} \int_0^1 B_n(x)\,dx \, \frac{z^n}{n!}.$$

$$\Rightarrow 1 = \int_0^1 B_0(x)\,dx + \sum_{n=1}^{\infty} \int_0^1 \frac{B_n(x)\,dx\, z^n}{n!}$$

$$\Rightarrow 1 = \int_0^1 1\,dx + \sum_{n=1}^{\infty} \int_0^1 \frac{B_n(x)\,dx\, z^n}{n!}$$

$$\Rightarrow 1 = 1 + \sum_{n=1}^{\infty} \int_0^1 \frac{B_n(x)\,dx\, z^n}{n!}$$

$$\Rightarrow \int_0^1 B_n(x)\,dx = 0 \quad \forall\, n \geq 1.$$

(ii) $$B_k(x) = \sum_{m=0}^{k} \binom{k}{m} B_m \, x^{k-m}$$

Note that

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z e^{xz}}{e^z-1}$$

$$= e^{xz} \cdot \frac{z}{e^z - 1}$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n z^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{B_m z^m}{m!} \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} \frac{B_m}{m!} \frac{x^{k-m}}{(k-m)!} \right) z^k$$

(CAUCHY PRODUCT) $\quad m+n=k$

$$= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} B_m x^{k-m} \binom{k}{m} \right) \frac{z^k}{k!}$$

Comparing the coefficients of the above two series, we complete the proof.

(iii) $$B_k(x+y) = \sum_{m=0}^{k} \binom{k}{m} B_m(x) y^{k-m}$$

Proof: Similar as (ii).

(iv) $B_n(1) = (-1)^n B_n$

Proof:
$$\sum_{n=0}^{\infty} \frac{B_n(1) z^n}{n!} = \frac{z e^z}{e^z - 1}$$

$$= \frac{z}{1-e^{-z}} = \frac{-z}{e^{-z}-1}$$

$$= \sum_{n=0}^{\infty} \frac{B_n (-z)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n B_n \frac{z^n}{n!}$$

(v) $B_n(1-x) = (-1)^n B_n(x)$

<u>Proof:</u> $\displaystyle\sum_{n=0}^{\infty} B_n(1-x) \frac{z^n}{n!} = \frac{z e^{(1-x)z}}{e^z - 1}$

$$= \frac{z e^z}{e^z - 1} \cdot e^{-xz}$$

$$= \frac{z e^{-xz}}{1-e^{-z}} = \frac{-z\, e^{x(-z)}}{e^{-z}-1}$$

$$= \sum_{n=0}^{\infty} \frac{B_n(x) (-z)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n B_n(x) \frac{z^n}{n!}.$$

Compare the coeff's to complete the proof.

$\boxtimes$

(vi) $B_n\left(\frac{1}{2}\right) = \left(2^{1-n} - 1\right) B_n$.

Proof: $$\sum_{n=0}^{\infty} \frac{B_n\left(\frac{1}{2}\right) z^n}{n!} = \frac{z e^{z/2}}{e^z - 1}$$

$$= \frac{z e^{z/2} + z - z}{e^z - 1}$$

$$= \frac{z\left(e^{z/2} + 1\right)}{e^z - 1} - \frac{z}{e^z - 1}$$

$$= \frac{z\,\cancel{\left(e^{z/2} + 1\right)}}{\cancel{\left(e^{z/2} + 1\right)}\left(e^{z/2} - 1\right)} - \frac{z}{e^z - 1}$$

$$= 2 \cdot \frac{z/2}{e^{z/2} - 1} - \frac{z}{e^z - 1}$$

$$= 2 \sum_{n=0}^{\infty} \frac{B_n (z/2)^n}{n!} - \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(2^{1-n} - 1\right) B_n \frac{z^n}{n!}.$$

Compare the coeff's to complete the proof. $\blacksquare$