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MA 631 - Special Functions - Lec. 3

(6) (Tutorial) $B_n(-x) = (-1)^n \left(B_n(x) + nx^{n-1}\right)$,

(7) $B_n(x+1) - B_n(x) = nx^{n-1}$.

Proof: LHS $= B_n(1-(-x)) - B_n(x)$

$$= (-1)^n B_n(-x) - B_n(x)$$
$$= (-1)^n \left\{ (-1)^n \left(B_n(x) + nx^{n-1}\right)\right\} - B_n(x)$$
$$= nx^{n-1}.$$

Remark: $B_n(x)$ is a solution of the difference equation
$$f(x+1) - f(x) = nx^{n-1}, \quad n \in \mathbb{N}\cup\{0\}.$$

(8) $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$.

Proof: Note that for $|z| < 2\pi$,
$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z e^{xz}}{e^z - 1}.$$

Since LHS converges uniformly on $[0, 1]$, we see that
$$\frac{d}{dx} \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{d}{dx} B_n(x)\right) \frac{z^n}{n!}.$$

On the other hand,

$$\frac{d}{dx}\,\frac{z e^{xz}}{e^{z}-1} = z\cdot \frac{z e^{xz}}{e^{z}-1}$$

$$= z \sum_{n=0}^{\infty} \frac{B_n(x)\, z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{B_n(x)\, z^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{B_{n-1}(x)\, z^n}{(n-1)!} = \sum_{n=1}^{\infty} n\, B_{n-1}(x)\, \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(n B_{n-1}(x)\right) \frac{z^n}{n!}.$$

$\blacksquare$

(9) $$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)\, x^n.$$

Proof: $(n+1)\, x^n \overset{(7)}{=} B_{n+1}(x+1) - B_{n+1}(x)$

$$= \sum_{m=0}^{n+1} \binom{n+1}{m} B_m(x) - B_{n+1}(x)$$

$$= \left\{ \sum_{m=0}^{n} \binom{n+1}{m} B_m(x) + B_{n+1}(x) \right\} - B_{n+1}(x)$$

$= \text{LHS}.$

$\blacksquare$

Remark:

Let $x=0$ in (9). Then

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} \binom{n+1}{k} B_k + \binom{n+1}{n} B_n = 0 .$$

$$\Rightarrow B_n = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k , \quad \left(\begin{array}{c}\text{Recursion} \\ \text{relation}\end{array}\right)$$

(10) $$\int_a^x B_n(t)\,dt = \frac{1}{n+1}\left(B_{n+1}(x) - B_{n+1}(a)\right).$$

Proof:

LHS $$\int_a^x \frac{1}{n+1} \frac{d}{dt} B_{n+1}(t)\,dt \qquad (\text{using } (8))$$

$$= \frac{1}{n+1} \Big[B_{n+1}(t)\Big]_a^x = \text{RHS}.$$

Remark: When $x = a+1$, we have

$$\int_a^{a+1} B_n(t)\,dt = \frac{1}{n+1}\left(B_{n+1}(a+1) - B_{n+1}(a)\right)$$

(7) $$= \frac{1}{n+1} \cdot (n+1) a^n = a^n .$$

$\longrightarrow$ (*)

Hence $S_n(m) = \sum_{j=0}^{n-1} j^m$

$$= \sum_{j=0}^{n-1} \int_j^{j+1} B_m(t)\,dt \quad (\text{using } \circledast)$$

$$= \int_0^n B_m(t)\,dt$$

$$= \frac{1}{m+1}\left(B_{m+1}(n) - B_{m+1}(0)\right)$$

$$= \frac{1}{m+1}\left(B_{m+1}(n) - B_{m+1}\right). \qquad \blacksquare$$

Fourier expansion of Bernoulli polynomials

Thm. 4

(i) For $0 < x < 1$, when $n = 0$ & for $0 \leq x \leq 1$ when $n > 0$, we have

$$B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{(2\pi m)^{2n+1}}$$

(ii) For $0 \leq x \leq 1$ and $n > 0$,

$$B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{\cos(2\pi m x)}{(2\pi m)^{2n}}.$$

Remark: Letting $x=0$ in (ii) gives

$$B_{2n} = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{1}{(2\pi m)^{2n}}$$

$$= \frac{2(-1)^{n+1}(2n)!}{(2\pi)^{2n}} \zeta(2n)$$

(where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $Re(s) > 1$, is the Riemann zeta function).

$$\Rightarrow \zeta(2n) = \frac{(2\pi)^{2n} B_{2n}}{2(-1)^{n+1}(2n)!} \quad \text{(Euler)}$$

Since π is transcendental, so is $\zeta(2n)$ for every $n \in \mathbb{N}$.

$$\Rightarrow \zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

$$\vdots$$

Ramanujan's formula for $\zeta(2n+1)$ states that if α, β are two number such that $Re(\alpha)>0$, $Re(\beta)>0$ & $\alpha\beta=\pi^2$, then for $n\neq 0$,

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{m^{-2n-1}}{e^{2m\alpha}-1}\right\}$$

$$=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{m=1}\frac{m^{-2n-1}}{e^{2m\beta}-1}\right\}$$

$$-2^{2n}\sum_{j=0}^{n+1}\frac{(-1)^j B_{2j}\, B_{2n+2-2j}}{(2j)!\,(2n+2-2j)!}\alpha^{n+1-j}\beta^j.$$

- We only know $\zeta(3)$ is irrational (Roger Apéry).
- At least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational (Wadim Zudilin).
- Lerch's formula: $\alpha=\beta=\pi$ & n odd,