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MA 631 - Special Functions - Lec. 4

Thm. 4

(i) For $0<x<1$, when $n=0$ & for $0\le x\le 1$ when $n>0$, we have

$$B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)!\sum_{k=1}^{\infty}\frac{\sin(2\pi k x)}{(2\pi k)^{2n+1}}$$

(ii) For $0\le x\le 1$ and $n>0$,

$$B_{2n}(x) = 2(-1)^{n+1}(2n)!\sum_{k=1}^{\infty}\frac{\cos(2\pi k x)}{(2\pi k)^{2n}}.$$

Proof: For pointwise convergence of the Fourier series of a function f, we need the function to be continuously diff. on the interval and it should agree at the end points -

In the case of $B_n(x)$, of course, it's cont. diff. on $[0,1]$ (since its a polynomial), and it agrees at the end points (except when $n=1$) because

1st proof: $B_n(1) = (-1)^n B_n$ (from Thm 3 (iv))

$$\Rightarrow B_{2n}(1) = B_{2n}$$

$$\& \quad B_{2n+1}(1) = (-1)^{2n+1}B_{2n+1} = -B_{2n+1} = 0$$

(for $n>0$) (from Thm. 3).

$$\Rightarrow \quad B_n(1) = B_n(0) \text{ (for } n > 1\text{)}$$

(Note: $B_1(x) = x - \frac{1}{2}$

$$\Rightarrow B_1(1) = \frac{1}{2} \neq -\frac{1}{2} = B_1 \text{)}$$

<u>2nd proof</u>: Note that $\int_0^1 B_n(x)\,dx = 0 \quad \forall n \in \mathbb{N}$.

$$\Rightarrow \quad 0 = \int_0^1 B_n(x)\,dx = \frac{1}{n+1} \int_0^1 \frac{d}{dx} B_{n+1}(x)\,dx$$

$$= \frac{1}{(n+1)} \left[B_{n+1}(1) - B_{n+1}(0) \right].$$

$$\Rightarrow \quad B_{n+1}(1) = B_{n+1}(0) \quad \forall n \in \mathbb{N},$$

Hence $B_n(x) = \displaystyle\sum_{k=-\infty}^{\infty} a_{n,k}\, e^{2\pi i k x}$,

where $a_{n,k} = \displaystyle\int_0^1 B_n(x)\, e^{-2\pi i k x}\,dx$

<u>Case I</u>: $n = 0$: Then $B_n(x) = B_0(x) = 1$.

Hence if $k = 0$, then

$$a_{0,0} = \int_0^1 dx = 1. \quad \text{——— } (1')$$

But if $k \neq 0$, then

$$a_{0,k} = \int_0^1 e^{-2\pi i k x}\, dx = \left[\frac{e^{-2\pi i k x}}{-2\pi i k}\right]_0^1$$

$$= \frac{e^{-2\pi i k} - e^{-2\pi i k(0)}}{-2\pi i k} = 0, \longrightarrow \text{(ii)}$$

<u>Case 2:</u> $n \neq 0 \Rightarrow n > 0$

If $k = 0$, then

$$a_{n,0} = \int_0^1 B_n(x)\, dx = 0 \quad \left(\text{from Thm. 3}\right)$$

If $k \neq 0$,

$$a_{n,k} = \int_0^1 B_n(x) e^{-2\pi i k x}\, dx$$

$$= \left[\frac{B_n(x) e^{-2\pi i k x}}{-2\pi i k}\right]_0^1 - \int_0^1 \left(\frac{d}{dx} B_n(x)\right) \frac{e^{-2\pi i k x}}{-2\pi i k}\, dx$$

$$= \frac{B_n(1) - B_n}{-2\pi i k} + \frac{1}{2\pi i k}\int_0^1 n B_{n-1}(x) e^{-2\pi i k x}\, dx$$

$$\Rightarrow a_{1,k} = \frac{B_1(1) - B_1}{-2\pi i k} + \frac{1}{2\pi i k}\int_0^1 e^{-2\pi i k x}\, dx$$

$$= \frac{\left(1-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)}{-2\pi i k} = \frac{-1}{2\pi i k} \quad \text{——— (iii)}$$

For $n \geq 1$,

$$a_{n,k} = 0 + \frac{n}{2\pi i k}\, a_{n-1,k}$$

$$= \left(\frac{1}{2\pi i k}\right)^2 n(n-1)\, a_{n-2,k}$$

$$= \left(\frac{1}{2\pi i k}\right)^3 n(n-1)(n-2)\, a_{n-3,k}$$

$$= \quad \vdots$$

$$= \left(\frac{1}{2\pi i k}\right)^{n-1} n(n-1)\cdots(n-(n-2))\, a_{1,k}$$

$$= \left(\frac{1}{2\pi i k}\right)^{n-1} n! \cdot \frac{-1}{2\pi i k}$$

$$= -\left(\frac{1}{2\pi i k}\right)^{n} n! = -\frac{(-i)^n}{(2\pi k)^n}\, n!$$

Hence for $n \geq 1$ & $k \neq 0$,

$$a_{n,k} = -\frac{(-i)^n}{(2\pi k)^n}\, n! \quad \text{——— (iv)}$$

$= 2m-1$

$$-(-i)^{2m-1} = -\frac{1}{-i}(-i)^{2m}$$

$$= \frac{1}{i}(-1)^m = \frac{1}{i}(-1)^{n+1/2}$$

Hence

$$a_{n,k} = \begin{cases} \dfrac{(-1)^{n/2+1}\, n!}{(2\pi k)^n}, & \text{if } n \geq 1 \text{ \& even} \\[2ex] \dfrac{-i\,(-1)^{\frac{n+1}{2}}\, n!}{(2\pi k)^n}, & \text{if } n \geq 1 \text{ \& odd} \\[2ex] 1, & n=0 \,\&\, k=0 \\ 0, & n=0,\ k \neq 0 \end{cases}$$

Hence

$$B_1(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_{1,k}\, e^{2\pi i k x}$$

$$= \sum_{k=-\infty}^{\infty} \frac{-1}{2\pi i k}\, e^{2\pi i k x}$$

$$= \frac{-1}{\pi} \sum_{k=1}^{\infty} \left(\frac{e^{2\pi i k x}}{k} + \frac{e^{-2\pi i k x}}{-k} \right) \cdot \frac{1}{2i}$$

$$= \frac{-1}{\pi} \sum_{k=1}^{\infty} \left(\frac{\;\frac{e^{2\pi i k x} - e^{-2\pi i k x}}{2i}\;}{k} \right)$$

$$\Rightarrow \quad x - \frac{1}{2} = -\sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{\pi k}$$

for $(0 < x < 1)$

For $n > 1$, n even:

$$B_n(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{n/2+1}\, n!}{(2\pi k)^n} \cdot e^{2\pi i k x}$$

$$= \frac{(-1)^{n/2+1}\, n!}{(2\pi)^n} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^n}$$

$$= \frac{(-1)^{n/2+1}\, n!}{(2\pi)^n} \left\{ \sum_{k=1}^{\infty} \frac{e^{2\pi i k x}}{k^n} + \frac{e^{-2\pi i k x}}{(-k)^n} \right\}$$

$$= \frac{(-1)^{n/2+1}\, n!}{(2\pi)^n} \sum_{k=1}^{\infty} \left(\frac{e^{2\pi i k x} + e^{-2\pi i k x}}{k^n} \right)$$

$$= \frac{2\,(-1)^{n/2+1}\, n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^n}.$$

$$\Rightarrow$$

$$B_{2n}(x) = 2\,(-1)^{n+1}(2n)! \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{(2\pi k)^{2n}}$$

· For $n > 1$, n odd:

$$B_n(x) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{-i(-1)^{\frac{n+1}{2}} n!}{(2\pi k)^n} e^{2\pi i k x}$$

$$= \frac{-i(-1)^{\frac{n+1}{2}} n!}{(2\pi)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^n}$$

$$= \frac{-i(-1)^{\frac{n+1}{2}} n!}{(2\pi)^n} \sum_{k=1}^{\infty} \left(\frac{e^{2\pi i k x}}{k^n} + \frac{e^{-2\pi i k x}}{(-k)^n} \right)$$

$$= \frac{-i(-1)^{\frac{n+1}{2}} n!}{(2\pi)^n} \sum_{k=1}^{\infty} \left(\frac{e^{2\pi i k x} - e^{-2\pi i k x}}{k^n} \right)$$

$$= 2(-1)^{\frac{n+1}{2}} n! \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{(2\pi k)^n}.$$

$\Longrightarrow$

$$B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{(2\pi k)^{2n+1}}$$

$\blacksquare$

Chapter 2 — The Gamma function

Motivation:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2} \quad (n \in \mathbb{N})$$

But $\frac{n(n+1)}{2}$ makes sense for any $n \in \mathbb{R}$ (well-defined). (or, for that matter, for any $n \in \mathbb{C}$).

Question: Can we extend

$1 \cdot 2 \cdot 3 \cdots n = n!$ to real values of n (or, for that matter, complex values of n)?

Euler found that

$$\int_0^\infty e^{-x} x^n \, dx = n! \quad \text{for } n \in \mathbb{N}.$$

Note that $\int_0^\infty e^{-x} x^{s-1} \, dx$ converges as long as $\mathrm{Re}(s) > 0$.

(See John B. Conway's 'Functions of one complex variables p. 180 — 182)

Def. $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \quad (\mathrm{Re}(s) > 0)$

(originally, Euler showed

$$n! = \int_0^1 (-\ln x)^n\, dx \;)$$

$$\Longrightarrow \quad \Gamma(n+1) = n!$$