19/11/22

MA 631 - Special Functions - Lec. 8

Thm. 11 (Duplication formula for $\Gamma(z)$) (Legendre)

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{2^{2z-1}}\Gamma(2z)$$

Proof: $B(z,z) = \int_0^1 t^{z-1}(1-t)^{z-1}\,dt$

for $\mathrm{Re}(z) > 0$. We remove this condition later, by analytic continuation.

$$= \int_0^1 \left(t(1-t)\right)^{z-1}\,dt$$

$$= 2\int_0^{1/2} \left(t(1-t)\right)^{z-1}\,dt$$

$$\left(\text{since } \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx\right.$$

$$\left. = 2\int_0^a f(x)\,dx, \quad \text{if } f(2a-x) = f(x)\right)$$

Let $s = 4t(1-t)$, or, in other words,

$$t = \frac{1-\sqrt{1-s}}{2} = \frac{1}{2} - \frac{\sqrt{1-s}}{2}$$

$$dt = \frac{1}{4\sqrt{1-s}}\,ds$$

when $t=0$, $s=0$; when $t=\frac{1}{2}$, $s=4\cdot\frac{1}{2}\cdot\frac{1}{2}=1$

$$\Rightarrow B(z,z) = 2\int_0^1 \left(\frac{s}{4}\right)^{z-1}\left(-\frac{1}{4\sqrt{1-s}}\right) ds$$

$$= 2^{1-2z+2}\int_0^1 s^{z-1}(1-s)^{\frac{1}{2}-1}\frac{ds}{4} \qquad \left(B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt\right)$$

$$= 2^{1-2z} B(z, \tfrac{1}{2})$$

$$\Rightarrow \frac{\cancel{\Gamma(z)}\Gamma(z)}{\Gamma(2z)} = 2^{1-2z}\frac{\cancel{\Gamma(z)}\Gamma(\frac{1}{2})}{\Gamma(z+\frac{1}{2})} \quad (\text{by Thm. 1})$$

$$\Rightarrow \Gamma(z)\Gamma(z+\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{2^{2z-1}}\Gamma(2z), \text{ for } Re(z)>0.$$

By analytic continuation, the formula holds for every $z\in\mathbb{C}$.

<u>2^{nd} proof</u> : We have shown in the prev. lecture that

$$B(p,q) = 2\int_0^{\pi/2}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta.$$

$$\Rightarrow B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2}\sin^p(\theta)\cos^q(\theta)\,d\theta$$

Let $p=q$. Then

$$B\left(\frac{p+1}{2}, \frac{p+1}{2}\right) = 2\int_0^{\pi/2}(\cos\theta\sin\theta)^p d\theta$$

$$= 2^{1-p}\int_0^{\pi/2} (\sin(2\theta))^p \, d\theta.$$

$$\left(\text{Let } 2\theta = t \quad d\theta = dt/2\right)$$

$$= 2^{-p}\int_0^{\pi} (\sin t)^p \, dt$$

$$= 2^{1-p}\int_0^{\pi/2} (\sin t)^p \, dt$$

$$= 2^{1-p}\int_0^{\pi/2} (\sin t)^p (\cos t)^0 \, dt$$

$$= \frac{2^{1-p}}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right)$$

$$\Rightarrow \frac{\cancel{\Gamma\left(\frac{p+1}{2}\right)}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)} = 2^{-p}\frac{\Gamma\left(\cancel{\frac{p+1}{2}}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}$$

Let $\frac{p+1}{2} = z \Rightarrow p = 2z - 1$

$$\Rightarrow \frac{\Gamma(z)}{\Gamma(2z)} = 2^{1-2z}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(z+\frac{1}{2}\right)}.$$

Thm. 12 (Reflection formula for $\Gamma(z)$)

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad (z \notin \mathbb{Z})$$

Remarks: 1 This shows that $\Gamma(z) \neq 0$ for any $z \in \mathbb{C}$. This is because:

RHS is never zero ($\because \sin(\pi z)$ is an entire function of z).

Suppose $\Gamma(z) = 0$ for some $z \in \mathbb{C}$ & $\Gamma(1-z) = \pm\infty$.
$\Rightarrow 1 - z = -n$, $n \in \mathbb{Z} \cup \{0\}$.
$\Rightarrow z = n+1$

$\Rightarrow \Gamma(z) = \Gamma(n+1) = n!,\ \neq 0$ (contradiction).

2 $z = \frac{1}{4}$. Then $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \dfrac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}$,

Proof: We assume the following result (to be proved in Tutorial. 2):

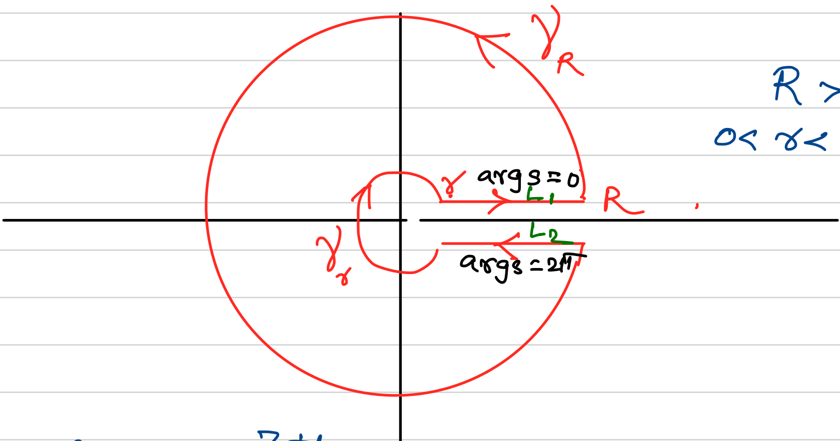
$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx \qquad \left(\begin{array}{l} Re(p) > 0, \\ Re(q) > 0 \end{array}\right)$$

We first prove the result for $0 < Re(z) < 1$, and later extend it by analytic continuation.

$$\Rightarrow \Gamma(z)\Gamma(1-z) = B(z, 1-z)$$

$$= \int_0^\infty \frac{x^{z-1}}{1+x}\, dx$$

Now consider $\int_0^\infty \frac{s^{z-1}}{1+s}\, ds$, $0 < \arg(s) < 2\pi$



$R > 1$, $0 < r < 1$;

Let $f(z) = \frac{s^{z+1}}{1+s}$.

f has a simple pole at $s = -1 = e^{\pi i}$.

$$\left[\int_{\gamma_R} + \int_{L_2} + \int_{\gamma_r} + \int_{L_1}\right] f(s)\, ds = 2\pi i \operatorname{Res}_{s=-1} f(z)$$

$$= 2\pi i \lim_{s\to -1} \frac{(s+1)s^{z-1}}{1+s} = 2\pi i\, e^{\pi i(z-1)}$$

$$= -2\pi i\, e^{\pi i z} \quad \text{———} \quad (A)$$

$$\left| \int_{\gamma_R} f(s)\,ds \right| \leq \int_0^{2\pi} \left| \frac{s^{z-1}}{1+s} \right| |ds|,$$

where $s = Re^{i\theta}$. Then $ds = iRe^{i\theta}\,d\theta$

$$= \int_0^{2\pi} \frac{|Re^{i\theta}|^{Re(z)-1}}{|1+Re^{i\theta}|} R\,d\theta$$

$$\leq \int_0^{2\pi} \frac{R^{Re(z)}}{R-1}\,d\theta$$

$\Big(\because |1+Re^{i\theta}| \geq ||1| - |Re^{i\theta}|| = R-1, \ (\because R>1)$ (actually R is large enough) $\Big)$

$$= \frac{R^{Re(z)}}{R-1} \int_0^{2\pi} d\theta = \frac{2\pi R^{Re(z)}}{R-1}.$$

$$= \frac{2\pi R^{Re(z)-1}}{1 - 1/R} \longrightarrow 0 \text{ as } R \to \infty$$

$(\because Re(z) < 1)$

Similarly,
$$\left| \int_{\gamma_r} \frac{s^{z-1}}{1+s}\,ds \right| \leq \int_{\gamma_r} \frac{|s^{z-1}|\,|ds|}{|1+s|}$$

$$\leq \frac{2\pi r^{Re(z)}}{1-r} \longrightarrow 0 \text{ as } r \to 0.$$

$(\because Re(z) > 0)$

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \left[\int_{\gamma_R} + \int_{\gamma_r} + \int_{L_1} + \int_{L_2} \right] f(s)\, ds = -2\pi i\, e^{\pi i z}$$

implies

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \left[\int_{L_1} + \int_{L_2} \right] f(s)\, ds = -2\pi i\, e^{\pi i z} \quad \text{——} \quad B$$