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MA 631 – SPECIAL FUNCTIONS – Lec. 9

Thm. 14 (Weierstrass product for $\Gamma(z)$)

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right], \text{ for } z \in \mathbb{C}.$$

Proof: $\prod_{k=1}^{\infty}(1+u_k)$ is said to be convergent if $\lim_{n\to\infty} P_n$ exists, when

$$P_n = \prod_{k=1}^{n}(1+u_k).$$

We know that [Conway's book, ch. 7] $\prod_{k=1}^{\infty}(1+u_k)$ converges iff $\sum_{k=1}^{\infty} \log(1+u_k)$ is convergent.

$(Re(u_k) > -1)$.

Claim: $\log\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right] = \log\left(1+\frac{z}{n}\right) - \frac{z}{n} = O_z\left(\frac{1}{n^2}\right)$

Notation: $f(x) = O(g(x))$ means $\exists$ constants x_0 and N such $\forall x \geq x_0$,

$$|f(x)| \leq N|g(x)|.$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} - \cdots$$

In our case

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| = \left|\sum_{m=1}^{\infty} \frac{(-1)^{m+1} z^m}{m\, n^m} - \frac{z}{n}\right|$$

$$\leq \sum_{m=2}^{\infty} \frac{|z/n|^m}{m} = \frac{|z|^2}{2n^2} + \sum_{m=3}^{\infty} \frac{|z/n|^m}{m}$$

$$= \frac{|z|^2}{2n^2} + \frac{|z|^2}{n^2} \sum_{m=1}^{\infty} \frac{|z/n|^m}{(m+2)}$$

$$\leq \frac{|z|^2}{2n^2} + \frac{|z|^2}{3n^2} \sum_{m=1}^{\infty} |z/n|^m \qquad \left(\because m \geq 1 \Rightarrow m+2 \geq 3 \Rightarrow \frac{1}{m+2} \leq \frac{1}{3}\right)$$

$$\leq \frac{C_z}{n^2}$$ ($\because$ z is fixed & we can choose n large enough so that $|z/n| < 1$)

$$\Rightarrow \log\left(1+\frac{z}{n}\right) - \frac{z}{n} = O_z\left(\frac{1}{n^2}\right)$$

So if $\log(1+u_n) = \log\left(1+\frac{z}{n}\right) - \frac{z}{n}$,

then $\sum_{n=1}^{\infty} \log(1+u_n)$ converges, and hence

$\prod_{n=1}^{\infty} \log(1+u_n)$, i.e., $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-z/n}$ converges too.

Note that

$$z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-z/n}\right]$$

$$= z \exp\left(z \lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\log m\right)\right)$$

$$\cdot \lim_{m\to\infty} \prod_{n=1}^{m} \left[\left(1+z/n\right) e^{-z/n}\right]$$

$$= z \lim_{m\to\infty} \left\{ e^{z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)-z\log m} \prod_{n=1}^{m}\left(1+\frac{z}{n}\right) e^{-z/n} \right\}$$

$$= z \lim_{m\to\infty} \left(m^{-z} \prod_{n=1}^{m}\left(1+\frac{z}{n}\right)\right) = z \lim_{m\to\infty}\left(m^{-z}\frac{(z+1)(z+2)\cdots(z+m)}{m!}\right)$$

Hence we will be done if we show that

$$\Gamma(z) = \lim_{m\to\infty} \frac{m^z\, m!}{z(z+1)(z+2)\cdots(z+m)}.$$

<u>Thm. 13 (Gauss)</u> For $z \neq 0, -1, -2, \cdots,$

$$\Gamma(z) = \lim_{m\to\infty} \frac{m^z\, m!}{z(z+1)(z+2)\cdots(z+m)}.$$

1^{st} proof: (For z real, such that $0 < z = x \leq 1$).

If f is convex on (a,b) and $a < s < t < u < b$, then

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$

(Ex. 23 on p. 101 of Rudin)

We know that $\log \Gamma(x)$ is convex on $(0,\infty)$.

Let $n \in \mathbb{N}$, $n \geq 2$.

Now $0 < x \leq 1$ implies

$$-1+n < n < n+x \leq n+1$$

Thus $n < x+n \leq n+1$ implies

$$\frac{\log \Gamma(x+n) - \log \Gamma(n)}{x} \leq \frac{\log \Gamma(n+1) - \log \Gamma(n)}{1} \quad \text{--- (i)}$$

Similarly, $-1+n < n < x+n$ gives

$$\frac{\log \Gamma(n) - \log \Gamma(n-1)}{1} \leq \frac{\log \Gamma(x+n) - \log \Gamma(n)}{x} \quad \text{--- (ii)}$$

From (i) & (ii),

$$\log\left(\frac{\Gamma(n)}{\Gamma(n-1)}\right) \leq \frac{\log(\Gamma(x+n)) - \log((n-1)!)}{x} \leq \log\left(\frac{\Gamma(n+1)}{\Gamma(n)}\right)$$

$$\Rightarrow \log(n-1) \leq \frac{\log(\Gamma(x+n)) - \log((n-1)!)}{x} \leq \log n.$$

$$\Rightarrow (n-1)^x \leq \frac{\Gamma(x+n)}{(n-1)!} \leq n^x$$

$$\Rightarrow (n-1)^x (n-1)! \leq \Gamma(x+n) \leq n^x (n-1)!$$

But $\Gamma(x+n) = (x+n-1)(x+n-2) \cdots (x+1)x\,\Gamma(x)$,

$$\Rightarrow \frac{(n-1)^x (n-1)!}{x(x+1)\cdots(x+n-1)} \leq \Gamma(x) \leq \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)}$$

Since the above inequalities hold $\forall n \geq 2$ we can replace n by $(n+1)$ on the extreme left, and have

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x) \leq \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)}$$

$$\Rightarrow \frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \cdot \frac{x+n}{n}$$

$$\frac{n}{x+n}\Gamma(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x)$$

Now let $n \to \infty$ so that

$$\Gamma(x) \leq \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x)$$

$$\Longrightarrow \quad \Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)} \quad \boxtimes$$

2nd proof: (For $z \in \mathbb{C}$). First, let $Re(z) > 0$.

Let $\Gamma(z,n) := \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$

Then

$$\lim_{n\to\infty} \Gamma(z,n) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$$

$$= \int_0^\infty \lim_{n\to\infty} \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt$$

(by integral analogue of Tannery's theorem: If $\lim_{N\to\infty} u(x,N) = v(x)$ and $\lim_{N\to\infty} \eta_N = \infty$, then

$$\lim_{N\to\infty} \int_a^{\eta_N} u(x,N)\, dx = \int_a^\infty v(x)\, dx,$$

provided $u(x,N) \to v(x)$ uniformly in any fixed interval, and $\exists$ a positive function $T(x) \ni |u(x,N)| \le T(x) \; \forall N$, where $\int_a^\infty T(x)\,dx$ converges.)