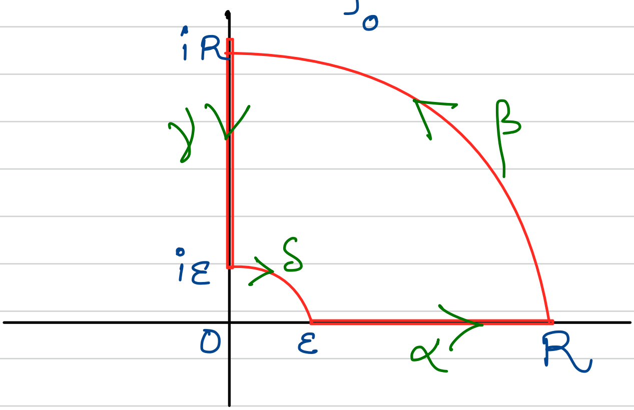
1/2/22

MA 631 – SPECIAL FUNCTIONS – Tut. 3

① $\int_0^\infty t^{z-1} f(t)\, dt$: Mellin transform of f :
(very useful of number theory)

Find $\int_0^\infty t^{z-1} \sin t \, dt$ & $\int_0^\infty t^{z-1} \cos t \, dt$
for $0 < \operatorname{Re}(z) < 1$.

We first evaluate $\int_0^\infty e^{-it}\, t^{z-1}\, dt$



By Cauchy's residue theorem,

$$\left[\int_\alpha + \int_\beta + \int_\gamma + \int_\delta \right] f(w)\, dw = 0,$$

where $f(w) = e^{-w} w^{z-1}$

Note that $\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\alpha f(w)\, dw = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R e^{-x} x^{z-1}\, dx$

$$= \int_0^\infty e^{-x} x^{z-1}\, dx = \Gamma(z) \qquad (\because \operatorname{Re}(z) > 0)$$

$$\text{——— (i)}$$

Evaluating $\int_\gamma f(w)\, dw$:

Let $w = it$. Then

$$\int_\gamma f(w)\, dw = i \int_R^\varepsilon e^{-it} (it)^{z-1}\, dt$$

$$= -i^z \int_\varepsilon^R e^{-it} t^{z-1}\, dt \qquad \text{——— (ii)}$$

Evaluating $\int_\beta f(w)\, dw$:

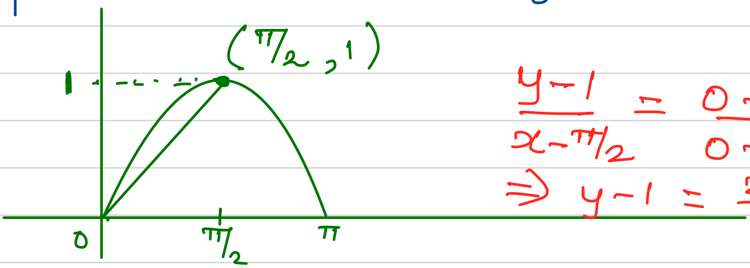
Let $w = Re^{i\theta}$ $\quad dw = iRe^{i\theta} d\theta$

$$\Rightarrow \int_\beta f(w)\, dw = \int_0^{\pi/2} e^{-Re^{i\theta}} (Re^{i\theta})^{z-1}\, iRe^{i\theta}\, d\theta$$

$$= iR^z \int_0^{\pi/2} e^{-Re^{i\theta}} (e^{i\theta})^z\, d\theta$$

$$= iR^z \int_0^{\pi/2} e^{-R\cos\theta - iR\sin\theta + i\theta z}\, d\theta$$

$$\Rightarrow \left|\int_\beta f(w)\,dw\right| \le R^{Re(z)} \int_0^{\pi/2} e^{-R\cos\theta - \theta\, Im(z)}\, d\theta \qquad \text{—— (a)}$$



$$\frac{y-1}{x-\pi/2} = \frac{0-1}{0-\pi/2} = \frac{2}{\pi}$$

$$\Rightarrow y-1 = \frac{2}{\pi}\left(x-\frac{\pi}{2}\right) = \frac{2x}{\pi} - 1$$

$\sin\theta \ge \frac{2\theta}{\pi}$ on $[0, \pi/2]$ JORDAN'S INEQUALITY

Let $\phi = \frac{\pi}{2} - \theta$. Then $\phi \in [0, \pi/2]$

$$\Rightarrow \sin\left(\frac{\pi}{2} - \phi\right) \ge \frac{2}{\pi}\left(\frac{\pi}{2} - \phi\right)$$

$$\Rightarrow \boxed{\cos\phi \ge 1 - \frac{2\phi}{\pi} \text{ on } [0, \pi/2]} \qquad \text{—— (b)}$$

Substituting (b) in (a), we get

$$\left|\int_\beta f(w)\,dw\right| \le R^{Re(z)} \int_0^{\pi/2} e^{-R\left(1-\frac{2\theta}{\pi}\right) - \theta\, Im(z)}\, d\theta$$

$$= R^{Re(z)} e^{-R} \int_0^{\pi/2} e^{\left(\frac{2R}{\pi} - Im(z)\right)\theta}\, d\theta$$

$$= R^{Re(z)} e^{-R} \left[\frac{e^{\left(\frac{2R}{\pi} - Im(z)\right)\theta}}{\frac{2R}{\pi} - Im(z)}\right]_0^{\pi/2}$$

$$= \frac{R^{Re(z)} e^{-R} \cdot \left(e^{R - \frac{\pi}{2} Im(z)} - 1 \right)}{\frac{2R}{\pi} - Im(z)}$$

$$= \frac{R^{Re(z)} \left(e^{-\frac{\pi}{2} Im(z)} - e^{-R} \right)}{\frac{2R}{\pi} - Im(z)}$$

$$\longrightarrow 0 \quad \text{as } R \to \infty . \qquad \text{—(iii)}$$

<u>Evaluating</u> $\int_\delta \underline{f(w)\,dw}$:

$$\left| \int_\delta f(w)\,dw \right| = \left| - \int_0^{\pi/2} e^{-\varepsilon e^{i\theta}} (\varepsilon e^{i\theta})^{z-1} i \varepsilon e^{i\theta} \, d\theta \right|$$

$$\leq \varepsilon^{Re(z)} \int_0^{\pi/2} e^{-\varepsilon\cos\theta - Im(z)\theta} \, d\theta$$

$$\leq \frac{\varepsilon^{Re(z)} \left(e^{-\frac{\pi}{2} Im(z)} - e^{-\varepsilon} \right)}{\frac{2\varepsilon}{\pi} - Im(z)}$$

as $\varepsilon \to 0$

$$\xrightarrow{\varepsilon \to 0} 0 \quad , \text{ if } Im(z) \neq 0$$

$$\lim_{\varepsilon \to 0} \varepsilon^{Re(z)} \cdot \frac{(1 - e^{-\varepsilon})}{\frac{2\varepsilon}{\pi}} \quad , \text{ if } Im(z) = 0 ,$$

$$\longrightarrow 0 \qquad \forall z \ni Re(z) > 0 , \qquad \text{—(iv)}$$

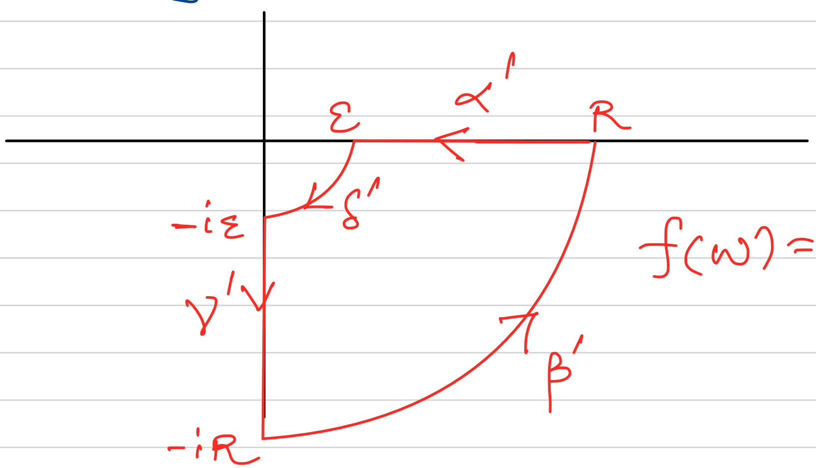
★ But $\lim_{\varepsilon \to 0} \dfrac{1-e^{-\varepsilon}}{2\varepsilon/\pi} = \lim_{\varepsilon \to 0} \dfrac{e^{-\varepsilon}}{2/\pi} = \dfrac{\pi}{2}$

Letting $\varepsilon \to 0$ & $R \to \infty$, & from (i)–(iv), we see that

$$-i^{z} \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} e^{-it} t^{z-1}\, dt = -\Gamma(z)$$

$$\Rightarrow \int_0^{\infty} e^{-it} t^{z-1}\, dt = \frac{\Gamma(z)}{i^{z}}. \qquad \text{— (A)}$$

Similarly by taking the contour



$f(w) = e^{-w} w^{z-1}$

we will find that

$$\int_0^{\infty} e^{it} t^{z-1}\, dt = \frac{\Gamma(z)}{(-i)^{z}} \qquad \text{— (B)}$$

$\frac{(B)+(A)}{2}$ gives

$$\int_0^\infty t^{z-1}\cos t\,dt = \frac{1}{2}\left\{\frac{\Gamma(z)}{(-i)^z} + \frac{\Gamma(z)}{i^z}\right\}$$

$$= \frac{1}{2}\Gamma(z)\left\{e^{\frac{i\pi z}{2}} + e^{-\frac{i\pi z}{2}}\right\}$$

$$= \Gamma(z)\cos\left(\frac{\pi z}{2}\right),$$

Similarly $\frac{(B)-(A)}{2i}$ gives

$$\int_0^\infty t^{z-1}\sin t\,dt = \Gamma(z)\sin\left(\frac{\pi z}{2}\right),$$

Remarks: ① $\int_0^\infty \frac{\sin(t)}{\sqrt{t}}\,dt$ exists

& is equals $\Gamma\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{\pi}}{\sqrt{2}}$

② Also note that

$$\int_0^\infty \underbrace{\frac{\sin t}{t}}_{\operatorname{sinc}(t)}\,dt = \frac{\pi}{2}$$

(doesn't come from our theorem)

$$\lim_{z \to 0} \Gamma(z) \sin\left(\frac{\pi z}{2}\right)$$

$$= \lim_{z \to 0} z\Gamma(z) \frac{\sin \frac{\pi z}{2}}{\frac{\pi z}{2}} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{2}.$$