MA 631: Special Functions (2022) - Tutorial 1

1. Prove that $B_n(-x) = (-1)^n \left(B_n(x) + nx^{n-1}\right)$.

2. For $m \geq 1$, prove Raabe's multiplication formula for Bernoulli polynomials:

$$\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) = m^{-n} B_n(mx).$$

3. (i) Prove that $\frac{z}{2}\left(\coth\left(\frac{z}{2}\right) - 1\right) = \frac{z}{e^z - 1}$.

(ii) Show that $2\coth(2z) - \coth(z) = \tanh(z)$ and hence show that for $|z| < \pi/2$,

$$\tanh(z) = 1 - \sum_{n=0}^{\infty} \frac{T_n z^n}{n!},$$

where T_n are the tangent numbers defined by

$$B_n = \frac{-nT_{n-1}}{2^n(2^n - 1)}, \quad n = 1, 2, \cdots,$$

and B_n are Bernoulli numbers.

(iii) Finally prove that for $|z| < \pi/2$,

$$\tan(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T_{2n+1} z^{2n+1}}{(2n+1)!}.$$