MA 631: Special Functions (2022) - Tutorial 2

1. Prove the following formulas for $n \in \mathbb{N}$:
$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{1-2n})B_{2n}}{2(2n)!},$$
$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{-2n})B_{2n}}{2(2n)!}.$$

2. If n is a positive integer and $a > b$, prove that
$$\int_0^{\pi} \frac{(\sin x)^{n-1}}{(a+b\cos x)^n}\,dx = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} \frac{(\Gamma(n/2))^2}{\Gamma(n)}.$$
(Hint: Note that
$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}, \qquad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.)$$

3. Prove that for $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$,
$$B(p,q) = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\,dx.$$

4. Prove the following functional relation
$$B(x,y) = \frac{x+y}{y} B(x, y+1).$$

5. Show that $\Gamma(x)$ is logarithmically convex on $(0, \infty)$.

Hint: Use Holder's inequality which states that if p and q are positive real numbers satisfying $1/p + 1/q = 1$ and f and g are non-negative Riemann integrable functions, then
$$\int_a^b fg\,dx \le \left(\int_a^b f^p\,dx\right)^{1/p} \left(\int_a^b g^q\,dx\right)^{1/q}. \tag{0.1}$$