MA 631: Special Functions - Tutorial 4 (2022)

1. For Re(c) >Re$(b) > 0$ and $|\arg(1-z)| < \pi$, prove Euler's transformation
$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). $$

2. In class, we have proved that
$$ {}_2F_1(a, b, a+b+1-c; 1-z) = P \cdot {}_2F_1(a, b; c; z) + Q \cdot z^{1-c} {}_2F_1(a-c+1, b-c+1; 2-c; z), $$
where $P = \dfrac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)}$. Show that $Q = \dfrac{\Gamma(c-1)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}$.

3. Show that
$$ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{L}} s^{-z} e^s \, ds, \qquad (z \in \mathbb{C}), $$
where the contour of integration $\mathcal{L}$ is the *Hankel contour* that runs from $-\infty$, $\arg s = -\pi$, encircles the origin in counter-clockwise direction, terminates at $-\infty$, now with $\arg s = \pi$.

4. Prove the following identities:
$$ \begin{aligned} {}_2F_1(1, 1; 2; z) &= -\frac{\log(1-z)}{z}, \\ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) &= \frac{1}{2z} \log\left(\frac{1+z}{1-z}\right), \\ {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) &= \frac{\tan^{-1}(z)}{z}, \\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) &= \frac{\sin^{-1}(z)}{z}, \\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) &= \frac{\log(z+\sqrt{1+z^2})}{z}. \end{aligned} $$