10/11/22

MA 631 – SPECIAL FUNCTIONS – Tut. 4

① $${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z).$$

for $Re(c) > Re(b) > 0$ & $|\arg(1-z)| < \pi$.

Proof: By Pfaff's transformation (Lec. 13),

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a & c-b \\ & c \end{matrix}; \frac{z}{z-1}\right)$$

for $Re(c) > Re(b) > 0$ & $|\arg(1-z)| < \pi$.

$$\Rightarrow {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = (1-z)^{-a} \left(1 - \frac{z}{z-1}\right)^{-(c-b)} \times {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix}; \frac{\frac{z}{z-1}}{\frac{z}{z-1} - 1}\right)$$

$$= (1-z)^{-a} \left(\frac{-1}{z-1}\right)^{b-c} {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix}; z\right)$$

$$\left(\frac{\frac{z}{z-1}}{\frac{z}{z-1} - 1} = \frac{z}{z-(z-1)} = z\right)$$

$$= (1-z)^{-a} (1-z)^{c-b}\, {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix}; z\right)$$

(2)

Find Q, where

$$ {}_2F_1\left(\begin{matrix} a & b \\ a+b+1-c \end{matrix}; 1-z\right) \qquad \text{—— } (*) $$

$$= P \cdot {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) + Q\, z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1, & b-c+1 \\ 2-c \end{matrix}; z\right),$$

where

$$P = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(b-c+1)\,\Gamma(a-c+1)}.$$

$$\mathrm{Re}(2-c-(a-c+1) - (b-c+1))$$
$$= \mathrm{Re}(\cancel{2}-\cancel{c}-a+\cancel{c}-\cancel{1} - b+c-\cancel{1})$$
$$= \mathrm{Re}(c-a-b) > 0$$

Ans. $Q = \dfrac{\Gamma(c-1)\,\Gamma(a+b+1-c)}{\Gamma(a)\,\Gamma(b)}$

<u>Proof:</u> We have initially assumed $\mathrm{Re}(c) < 1$ & $\mathrm{Re}(c-a-b) > 0$.

Let $z=1$ in $(*)$ so that

$$1 = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(b-c+1)\,\Gamma(a-c+1)} \cdot \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$
$$+ Q\, \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(2-c-a+c-1)\,\Gamma(2-c-b+c-1)}$$

$$\Rightarrow 1 = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)\,\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(b-c+1)\,\Gamma(a-c+1)\,\Gamma(c-a)\,\Gamma(c-b)}$$
$$+ Q\, \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}.$$

$$\Rightarrow 1 - \frac{\dfrac{\cancel{\pi}}{\sin(\pi c)} \cdot \dfrac{\cancel{\pi}}{\sin \pi(c-a-b)}}{\dfrac{\cancel{\pi}}{\sin \pi(c-a)} \cdot \dfrac{\cancel{\pi}}{\sin \pi(c-b)}}$$

$$= Q \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}.$$

$$\Rightarrow \frac{\sin(\pi c)\sin \pi(c-a-b) - \sin(\pi(c-a))\sin(\pi(c-b))}{\sin(\pi c)\sin(\pi(c-a-b))}$$

$$= Q \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}$$

$$\Rightarrow \frac{\cos(\pi(a+b)) - \cos(\pi(\cancel{2c}-a-b)) - \left\{\cos(\pi(b-a)) - \cos(\pi(\cancel{2c}-a-b))\right\}}{2\sin(\pi c)\sin(\pi(c-a-b))}$$

$$= Q \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}.$$

$$\Rightarrow \frac{2\sin\left(\dfrac{\pi(a+b)+\pi(b-a)}{2}\right)\sin\left(\dfrac{\pi(b-a)-\pi(a+b)}{2}\right)}{2\sin(\pi c)\sin(\pi(c-a-b))}$$

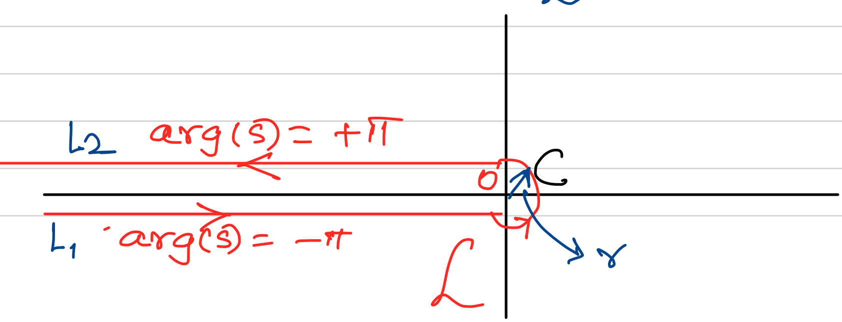
$$= Q \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}$$

$$\Rightarrow \frac{-2\sin(\pi b)\sin(\pi a)}{2\sin(\pi c)\sin(\pi(c-a-b))} = \varphi\,\frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}$$

$$\Rightarrow -\frac{\Gamma(c)\Gamma(1-c)\,\cancel{\Gamma(c-a-b)}\,\Gamma(1-c+a+b)}{\Gamma(b)\,\cancel{\Gamma(1-b)}\,\Gamma(a)\,\cancel{\Gamma(1-a)}} = \varphi\,\frac{\Gamma(2-c)\,\cancel{\Gamma(c-a-b)}}{\cancel{\Gamma(1-a)}\,\cancel{\Gamma(1-b)}}$$

$$\Rightarrow \varphi = -\frac{\Gamma(c)}{(1-c)}\,\frac{\Gamma(1-c+a+b)}{\Gamma(a)\,\Gamma(b)} = \frac{\Gamma(c-1)\,\Gamma(a+b-c+1)}{\Gamma(a)\,\Gamma(b)}.$$

(3) $$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{L}} s^{-z} e^{s}\,ds \qquad (z\in\mathbb{C})$$



Note that $s^{-z} = e^{-z\log s}$ & so we have to avoid $s \in \{w \in \mathbb{R} ; w \leq 0\}$.

We first assume $Re(z) < 1$.

(a)

Contribution of the circle C (of radius r):

$$\int_C s^{-z} e^{s}\, ds$$

Let $s = re^{i\theta}$ $\quad ds = ire^{i\theta} d\theta$

$$\left| \int_C s^{-z} e^{s}\, ds \right| \leq \int_C |s^{-z} e^{s}|\, |ds|$$

$$|s^{-z} e^{s}| = \left| (re^{i\theta})^{-z} e^{re^{i\theta}} \right|$$

$$= r^{-Re(z)} e^{\theta Im(z) + r\cos\theta}.$$

$$\Rightarrow \left| \int_C s^{-z} e^{s}\, ds \right| \leq \int_0^{2\pi} r^{1-Re(z)} e^{\theta Im(z) + r\cos\theta}\, d\theta$$

$$\leq r^{1-Re(z)} e^{r} \left. \frac{e^{\theta Im(z)}}{Im(z)} \right|_0^{2\pi}$$

$$= e^{r} r^{1-Re(z)} \frac{e^{2\pi Im(z)} - 1}{Im(z)}$$

$\longrightarrow 0$ as $r \to 0$ ($\because Re(z) < 1$).