13/2/22

MA 631 – SPECIAL FUNCTIONS – Tut-5

Quiz 1 problems

① Riemann ξ – function:

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

ⓐ $\xi(1-s) = \xi(s)$.

$$\xi(1-s) = \frac{(1-s)(1-s-1)}{2} \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

$= \xi(s)$ (using the functional eqn. for $\zeta(s)$).

ⓑ $\xi(s)$ is an entire fn. of s.

$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is analytic in $\mathbb{C} \setminus \{0, 1\}$

(see the lecture on funct. eqn. of $\zeta(s)$).

Hence mult. it by $\frac{s(s-1)}{2}$ makes it entire, as the poles of $\Gamma\left(\frac{s}{2}\right)$ & $\zeta(s)$ at $s = 0, 1$ resp. are killed by the zeros of $s(s-1)$ at $s = 0, 1$ resp.

Hence $\xi(s)$ is entire.

② $$\frac{2}{e^t+1} = 1 - \sum_{k=1}^{\infty} 2(2^{2k}-1)\frac{B_{2k}}{(2k)!}t^{2k-1} \quad (|t|<\pi)$$

$\parallel$

① $$\frac{2e^{-t/2}}{e^{t/2}+e^{-t/2}} = 1 - \tanh\left(\frac{t}{2}\right)$$

$$1 - \frac{2}{e^t+1} = \tanh(t)$$

Another proof:

$$\frac{2}{e^t+1} = \frac{2(e^t-1)}{e^{2t}-1} = \frac{2e^t}{e^{2t}-1} - \frac{2}{e^{2t}-1}$$

$$= \frac{1}{t}\left(\frac{2te^{\frac{1}{2}(2t)}}{e^{2t}-1} - \frac{2te^{0(2t)}}{e^{2t}-1}\right)$$

[since

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \quad (|t|<2\pi)$$

implies

$$\frac{2te^{2t(\frac{1}{2})}}{e^{2t}-1} = \sum_{n=0}^{\infty} \frac{B_n(1/2)(2t)^n}{n!} \quad (|t|<\pi)$$

$$= \frac{1}{t}\left(\sum_{n=0}^{\infty} \frac{B_n\left(\frac{1}{2}\right)(2t)^n}{n!} - \sum_{n=0}^{\infty} \frac{B_n(0)(2t)^n}{n!}\right)$$

$$= \frac{1}{t}\left(\sum_{n=1}^{\infty} \frac{B_n\left(\frac{1}{2}\right)(2t)^n}{n!} - \sum_{n=1}^{\infty} \frac{B_n(0)(2t)^n}{n!}\right)$$

($\because$ $n=0$ terms of both series cancel out)

$$= \sum_{n=1}^{\infty} \frac{(2^{1-n}-1)B_n 2^n t^{n-1}}{n!} - \sum_{n=1}^{\infty} \frac{B_n 2^n t^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{2^n t^{n-1}}{n!} B_n (2^{1-n} - 2)$$

$$= -2B_1 + \sum_{n=1}^{\infty} \frac{2^{2n} t^{2n-1} B_{2n} (2^{1-2n} - 2)}{(2n)!}$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{(1-2^{2n})}{(2n)!} t^{2n-1} B_{2n}$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{(2^{2n}-1) B_{2n} t^{2n-1}}{(2n)!}$$

Homework 1

(*) For $Re(\log a) > 0$ & $Re(a) > -1$, evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx$.

Convergence of the integral:

As $x \to 0^+$: $\frac{x^a}{a^x}$ behaves like x^a,

so to secure convergence at 0, we need $Re(a+1) > 0$, i.e., $Re(a) > -1$,

As $x \to \infty$: $\frac{x^a}{a^x} = e^{-x\log a}\,x^a$,

so to secure convergence at ∞, we need $Re(\log a) > 0$,

First, let $a \in \mathbb{R} \ni a > 1$. Then $\log a > 0$.

Let $a^x = e^t \implies x\log a = t$

$\implies dx = \frac{dt}{\log a}$.

when $x = 0$, $t = 0$

when $x = \infty$, $t = \infty$.

Hence

$$\int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^{(a+1)-1} \, dt$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}. \qquad \mathrm{Re}(a+1) > 0$$

Hence

$$\int_0^\infty \frac{x^a}{a^x} \, dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}, \quad \text{for } a > 1 \_\_ (\S)$$

We now show that both sides are analytic in $\mathrm{Re}(a) > -1$ & $\mathrm{Re}(\log a) > 0$. Then by identity thm., $(\S)$ is valid in this region.

Analyticity of LHS of $(\S)$ is guaranteed by Thm. $(*)$ of Lec. 6.

Analyticity of RHS of $(\S)$:

$\Gamma(a+1)$ is analytic in $\mathrm{Re}(a) > -1$.

$$\frac{1}{(\log a)^{a+1}} = e^{-(a+1)\log \log a}$$

This expression is analytic as long as $a \notin (-\infty, 0]$ and $\log a \notin (-\infty, 0]$.

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$a \notin (-\infty, 0] \qquad\qquad a \notin (0, 1]$$

$\Rightarrow a \notin (-\infty, 1]$

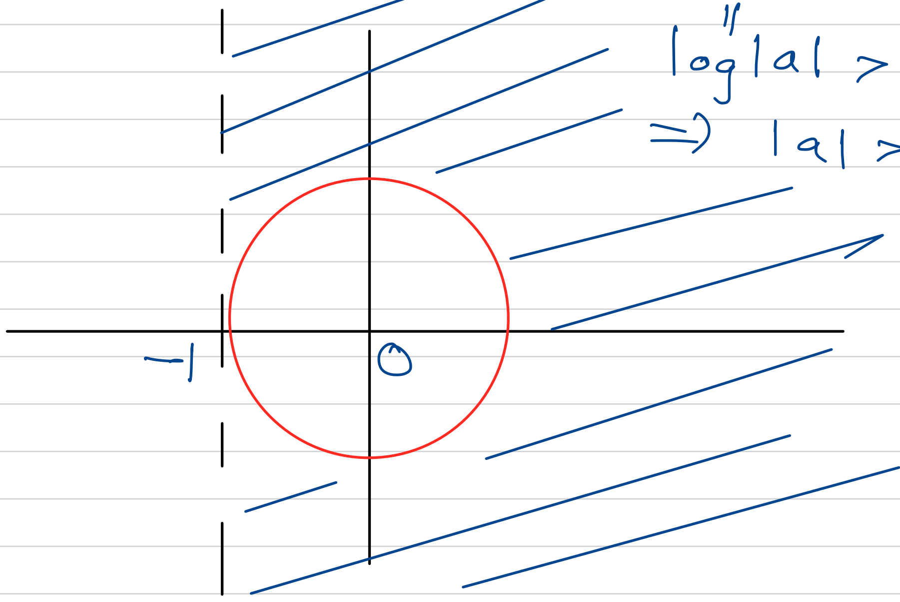
(Re(a) > −1 was anyway implying that $a \notin [-\infty\ -1]$, but what we have shown above also tells us that $a \notin (-1, 1]$.

Remark: That $|a| \nleq 1$ for a real can also be seen from ~~the fact~~ $\mathrm{Re}(\log a) > 0$

$$\Updownarrow$$

$$\log|a| > 0$$

$$\Rightarrow |a| > 1$$



Problem
$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}.$$

Proof: Let $\sin^2\phi = \frac{1}{2}\sin^2\theta$

$$\sin\phi = \frac{1}{\sqrt{2}}\sin\theta$$

$$\cos\phi \, d\phi = \frac{1}{\sqrt{2}}\cos\theta \, d\theta.$$

$$d\theta = \frac{\sqrt{2}\cos\phi}{\cos\theta} d\phi$$

$$= \frac{\sqrt{2}\cos\phi}{\sqrt{1-\sin^2\theta}} d\phi \quad = \frac{\sqrt{2}\cos\phi}{\sqrt{1-2\sin^2\phi}}$$

$$= \frac{\sqrt{2}\cos\phi}{\sqrt{\cos 2\phi}} d\phi$$

when $\theta = 0 \quad \phi = 0$

when $\theta = \pi/2$, $\phi = \frac{\pi}{4}$

$$\Rightarrow \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \int_0^{\pi/4} \frac{1}{\cos\phi} \cdot \frac{\sqrt{2}\cos\phi \, d\phi}{\sqrt{\cos 2\phi}}$$

$$= \sqrt{2}\int_0^{\pi/4} (\cos 2\phi)^{-1/2} d\phi$$

Let $2\phi = t \qquad 2d\phi = dt$

$$= \frac{\sqrt{2}}{2}\int_0^{\pi/2} (\cos t)^{-1/2} dt$$

Now $\int_0^{\pi/2} (\sin\theta)^p (\cos\theta)^q \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\Gamma^2\left(1/4\right)}{\frac{\pi}{\sin\left(\pi/4\right)}}$$

$$= \frac{\Gamma^2\left(1/4\right)}{4\sqrt{\pi}}$$