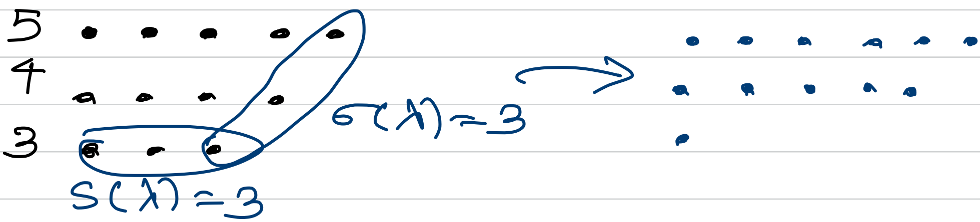
18/8/21

# MA 633 – Partition Theory – Lec. 10

Case 1 breaks down precisely when
$\sigma(\lambda) = r$, $s(\lambda) = r$.




In this case, the number being partitioned is $r + (r+1) + \cdots + (r + (r-1))$ —— (**)

$$= r^2 + (1 + 2 + \cdots + r - 1)$$
$$= r^2 + \frac{r(r-1)}{2} = \frac{3r^2 - r}{2} = \frac{r(3r-1)}{2}.$$

Consequently, if $n$ is a generalized pentagonal number, say, $n = \frac{r(3r \pm 1)}{2}$, then $p_e(\mathcal{D}, n) = p_o(\mathcal{D}, n) + (-1)^r$,
otherwise, $p_o(\mathcal{D}, n) = p_e(\mathcal{D}, n)$

Note that the partitions in (*) & (**) are unique.

Also, suppose $r \neq m$ & $\frac{r(3r-1)}{2} = \frac{m(3m+1)}{2}$,
then $r - m = -\frac{1}{3} \longrightarrow$ contradiction.

or if $r = m$, i.e; $\frac{r(3r-1)}{2} = \frac{r(3r+1)}{2}$,

then we get $1 = -1$ ~~→~~ ✗.

## Partitions: Yesterday and Today (by George Andrews)

**Thm. 16** Let $p(\text{"}H\text{"}, m, n)$ be the number of partitions of $n$ into $m$ parts coming from set $H$. Then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(\text{"}H\text{"}, m, n) z^m q^n = \prod_{n \in H} \frac{1}{(1 - zq^n)}$$

**Proof:** The proof can be made rigorous by first restricting $H$ to be a finite set, and then passing to the limit, provided $|q| < 1$, & $|zq| < 1$.

$$\prod_{n \in H} \frac{1}{1 - zq^n} = \prod_{n_i \in H} \sum_{m_i = 0}^{\infty} z^{m_i} q^{m_i n_i}$$

$$= \sum_{m_1, m_2, \dots = 0}^{\infty} z^{m_1 + m_2 + \cdots} q^{m_1 n_1 + m_2 n_2 + \cdots}$$

$$= \sum_{n=0}^{\infty} p(\text{"}H\text{"}, m, n) z^m q^n.$$

Similarly one can prove

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_d(\text{"}H\text{"}, m, n)\, z^m q^n = \prod_{n\in H}(1+zq^n)$$

Proof: $\prod_{n\in H}(1+zq^n) = \prod_{n_i\in H}\sum_{m_i=0}^{1} z^{m_i} q^{m_i n_i}$ ∎

Thm. 17 Let $d_s(n)$ denote the number of partitions of $n$ into exactly $s$ distinct parts. Then

$$\sum_{n=0}^{\infty} d_s(n) q^n = \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s}.$$

6

5+1

4+2

4+1+1

3+3

3+2+1

3+1+1+1

2+2+2

2+2+1+1

2+1+1+1+1

1+1+1+1+1+1

$q^6$

$$\sum_{s=0}^{\infty} \frac{(-z)^s q^{\frac{s(s-1)}{2}}}{(q;q)_s} = (z;q)_\infty$$