25/8/21

MA 633 – Partition Theory – Lec. 11

Thm. 17 Let $d_s(n)$ denote the number of partitions of n into exactly s distinct parts. Then

$$\sum_{n=0}^{\infty} d_s(n) q^n = \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s} .$$

Proof: Let $\pi : b_s + b_{s-1} + \cdots + b_2 + b_1$ be a partition enumerated by $d_s(n)$.

Then $b_1 < b_2 < b_3 < \cdots < b_{s-1} < b_s$.

Let $a_i = b_i - i$. Then

$$0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{s-1} \leq a_s .$$

(Note that $b_1 \geq 1$, $b_2 \geq 2$,

$$\begin{aligned} a_2 - a_1 &= (b_2 - 2) - (b_1 - 1) \\ &= b_2 - b_1 - 1 \\ &\geq 0 \quad (\because b_2 \geq b_1 + 1) \end{aligned}$$

Do this for other consecutive pairs.)

$$\sum_{n=0}^{\infty} d_s(n) q^n = \sum_{n=0}^{\infty} \sum_{\substack{\pi \vdash n \\ \pi = (b_s, b_{s-1}, \ldots, b_1)}} q^{b_1 + b_2 + \cdots + b_s}$$

$$= \sum_{a_1=0}^{\infty} \sum_{a_2=a_1}^{\infty} \sum_{a_3=a_2}^{\infty} \cdots \sum_{a_s=a_{s-1}}^{\infty} q^{(1+a_1)+(2+a_2)+\cdots(s+a_s)}$$

$$= q^{\frac{s(s+1)}{2}} \sum_{a_1=0}^{\infty} \sum_{a_2=a_1}^{\infty} \cdots \sum_{a_{s-1}=a_{s-2}}^{\infty} \sum_{a_s=a_{s-1}}^{\infty} q^{a_1+a_2+\cdots+a_s}$$

$$= q^{\frac{s(s+1)}{2}} \sum_{a_1=0}^{\infty} \sum_{a_2=a_1}^{\infty} \cdots \sum_{a_{s-1}=a_{s-2}}^{\infty} q^{a_1+a_2+\cdots+a_{s-1}} \sum_{a_s=a_{s-1}}^{\infty} q^{a_s}$$

Now $\quad a_s = a_{s-1} + m$

$$\sum_{a_s=a_{s-1}}^{\infty} q^{a_s} = \sum_{m=0}^{\infty} q^{a_{s-1}+m} = q^{a_{s-1}} \sum_{m=0}^{\infty} q^{m}$$

$$= \frac{q^{a_{s-1}}}{1-q}$$

Hence

$$\sum_{n=0}^{\infty} d_s(n) q^n$$

$$= \frac{q^{\frac{s(s+1)}{2}}}{(1-q)} \sum_{a_1=0}^{\infty} \sum_{a_2=a_1}^{\infty} \cdots q^{a_1+\cdots+a_{s-2}} \sum_{a_{s-1}=a_{s-2}}^{\infty} q^{2a_{s-1}}$$

Now let $\quad a_{s-1} = a_{s-2} + n$

$$\sum_{a_{s-1}=a_{s-2}}^{\infty} q^{2a_{s-1}} = \sum_{n=0}^{\infty} q^{2(a_{s-2}+n)}$$

$$= \frac{q^{2a_{s-2}}}{(1-q^2)}$$

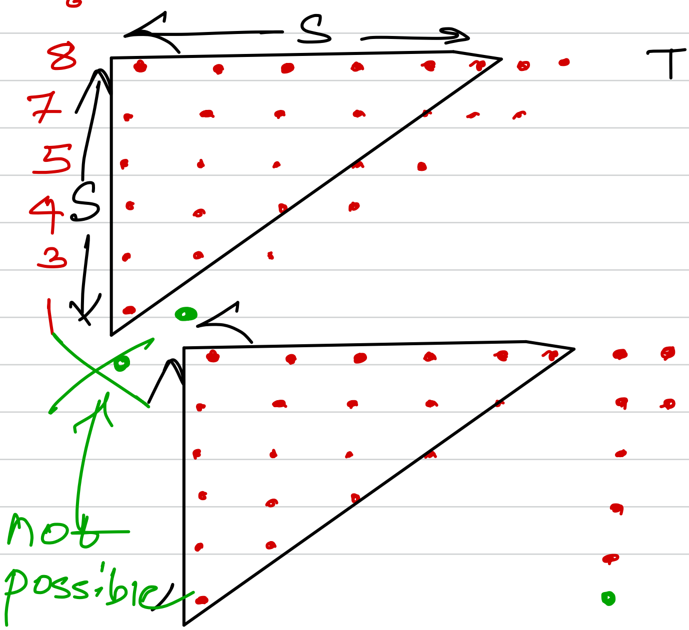
Iterating this process, we see that

$$\sum_{n=0}^{\infty} d_s(n) q^n = \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s}$$

$\boxtimes$

Combinatorial proof:

Consider a partition below enumerated by $d_6(n)$: (here $s=6$)



The Δ constitutes $\frac{s(s+1)}{2}$ nodes

Hence the generating function of the number of partitions of the remaining number $\left(n - \frac{s(s+1)}{2}\right)$ is that of the no. of ptns. into parts not exceeding s (by conjugation) & is given by

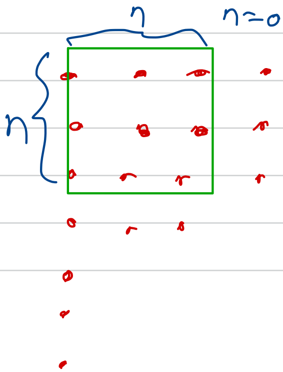
$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^s)}$$

Hence the reqd. gen. fn. is $\dfrac{q^{\frac{s(s+1)}{2}}}{(q;q)_s}$.

(*) If we sum over all s, we will get the g.f. for the number of partitions of n into distinct parts, i.e. $(-q;q)_\infty$.

Hence $\displaystyle\sum_{s=0}^{\infty} \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s} = (-q;q)_\infty$, which can be derived by letting $z=-q$ in $\displaystyle\sum_{n=0}^{\infty} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = (z)_\infty$.

Thm. 18 $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}$



Proof:

The generating function for all partitions with Durfee square of side n is

$$\sum_{N=0}^{\infty} q^N \sum_{\substack{n^2+m+k=N \\ m,k\geq 0}} p(\{1,2,\ldots,n\}, m)\, p(\{1,2,\ldots,n\}, k)$$

$$= \sum_{m,k\geq 0} q^{n^2+m+k}\, p(\{1,2,\ldots,n\}\, m)\, p(\{1,2,\ldots,n\}, k)$$

$$= q^{n^2} \left(\sum_{m=0}^{\infty} p(\{1,2,\ldots,n\}, m)\, q^m \right) \left(\sum_{k=0}^{\infty} p(\{1,2,\ldots,n\}, k)\, q^k \right)$$

$$= q^{n^2} \cdot \frac{1}{(q;q)_n} \frac{1}{(q;q)_n} = \frac{q^{n^2}}{(q;q)_n^2}.$$

$\boxtimes$

Next time:

Thm. 19 (Cauchy)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n (zq;q)_n} = \frac{1}{(zq;q)_\infty}.$$