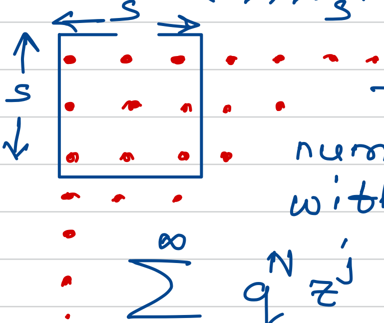
26/8/21

MA 633 - Partition Theory - Lec. 12

Thm. 19 (Cauchy)

$$\sum_{s=0}^{\infty} \frac{z^s q^{s^2}}{(q;q)_s (zq;q)_s} = \frac{1}{(zq;q)_\infty}.$$



The generating function of the number of partitions into j parts with Durfee square of side s is

$$\sum_{\substack{N=0 \\ j\geq s}}^{\infty} q^N z^j \sum_{s^2+m+n=N} p(\{1,2,3,\dots,s\}, j-s, m) \times p(\{1,2,3,\dots,s\}, n)$$

$$= z^s q^{s^2} \left(\sum_{\substack{m\geq 0 \\ j\geq s}} p(\{1,2,\dots,s\}, j-s, m)\, z^{j-s} q^m \right)$$

$$\times \left(\sum_{n\geq 0} p(\{1,2,\dots,s\}, n)\, q^n \right)$$

$$= z^s q^{s^2} \left(\sum_{\substack{m\geq 0 \\ j\geq 0}} p(\{1,2,\dots,s\}, j, m)\, z^j q^m \right)$$

$$\times \left(\sum_{n\geq 0} p(\{1,\dots,s\}, n)\, q^n \right)$$

$$= z^s q^{s^2} \frac{1}{(1-zq)(1-zq^2)\cdots(1-zq^s)} \cdot \frac{1}{(1-q)\cdots(1-q^s)}$$

$$= \frac{z^s q^{s^2}}{(zq;q)_s\,(q;q)_s}.$$

Hence summing over all s (from 0 to ∞) gives

$$\sum_{s=0}^{\infty} \frac{z^s q^{s^2}}{(zq;q)_s\,(q;q)_\infty} = \frac{1}{(zq;q)_\infty}.$$

<u>Question</u> : Can one find an identity for $(-zq;q)_\infty$ from Durfee square analysis of partitions into distinct parts ?

<u>Thm. 20</u>

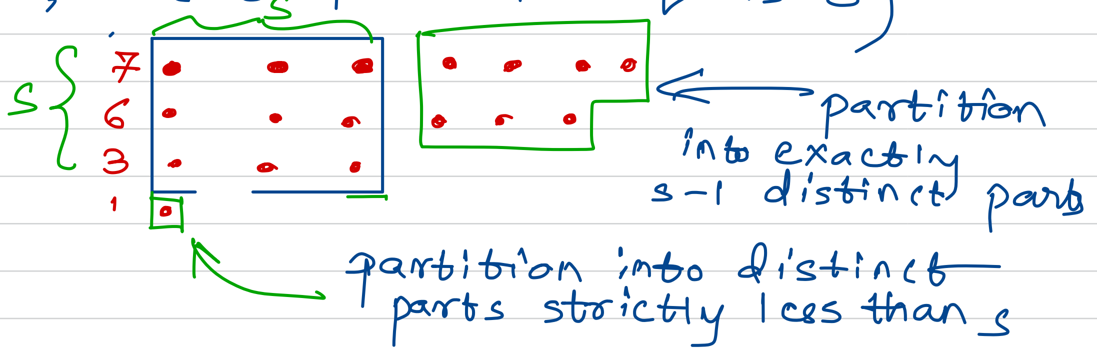
$$\sum_{s=0}^{\infty} z^s q^{\frac{3s^2+s}{2}} \frac{(-zq;q)_s}{(q;q)_s}\left(1+zq^{2s+1}\right)$$

$$= (-zq;q)_\infty$$

$$(1+zq)(1+zq^2)(1+zq^3)\cdots$$

Proof: Suppose π is a partition into distinct parts and Durfee square of side s.

<u>Case 1</u>: The lower edge of the Durfee square constitutes a complete part of π. (i.e; the s^{th} part of π equals s)



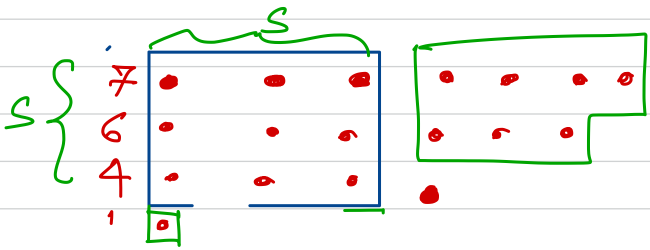
$$\sum_{N=0}^{\infty}\sum_{M=0}^{\infty} z^M q^N \sum_{s^2+n+m=N}\ \sum_{s+w=M} P_d(\{1,2,\ldots,s-1\},w,n) \times d_{s-1}(m)$$

$$= z^s q^{s^2}\left(\sum_{n=0}^{\infty} P_d(\{1,2,\ldots,s-1\},w,n)\, z^w q^n\right) \times \left(\sum_{m=0}^{\infty} d_{s-1}(m)\, q^m\right)$$

$$= z^s q^{s^2}(1+zq)(1+zq^2)\cdots(1+zq^{s-1}) \times \frac{q^{\frac{s(s-1)}{2}}}{(q;q)_{s-1}}$$

$$= z^s q^{s^2} (-zq;q)_{s-1} \frac{q^{\frac{s(s-1)}{2}}}{(q;q)_{s-1}} .$$

Case 2 Lower edge of the D.S. doesn't constitute a complete part of π.



partition into exactly s distinct parts

partition into distinct parts less than or equal to s

So the corresponding g.f. here is

$$z^s q^{s^2} (-zq;q)_s \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s} .$$

Hence we finally have

$$1 + \sum_{s=1}^{\infty} z^s q^{s^2} (-zq;q)_{s-1} \frac{q^{\frac{s(s-1)}{2}}}{(q;q)_{s-1}}$$

$$+ \sum_{s=1}^{\infty} z^s q^{s^2} (-zq;q)_s \frac{q^{\frac{s(s+1)}{2}}}{(q;q)_s} = (-zq;q)_\infty$$

$\Rightarrow (-zq;q)_\infty$

$$= 1 + \sum_{s=0}^{\infty} z^{s+1} q^{s^2+2s+1+\frac{s^2+s}{2}} \frac{(-zq;q)_s}{(q;q)_s}$$

$$+ \sum_{s=1}^{\infty} z^s q^{s^2} (-zq;q)_s \frac{q^{\frac{s^2+s}{2}}}{(q;q)_s}$$

$$= \sum_{s=0}^{\infty} z^{s+1} q^{\frac{3s^2+5s+2}{2}} \frac{(-zq;q)_s}{(q;q)_s}$$

$$+ \sum_{s=0}^{\infty} z^s q^{\frac{3s^2+s}{2}} \frac{(-zq;q)_s}{(q;q)_s}$$

$\Rightarrow$

$$\boxed{\begin{aligned}&\sum_{s=0}^{\infty} z^s q^{\frac{3s^2+s}{2}} \frac{(-zq;q)_s}{(q;q)_s} (1+zq^{2s+1}) \\ &= (-zq;q)_\infty\end{aligned}}$$

g.f. for ptns. into distinct parts with z keeping track of the number of parts.

2.1 Cor. (Euler's PNT)

Proof. Let $z=-1$ in the above theorem. Then

$$(q;q)_\infty = \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(3s+1)}{2}} (1-q^{2s+1})$$

$$= \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(3s+1)}{2}} - \sum_{s=0}^{\infty} (-1)^s q^{\frac{3s^2+5s+2}{2}}$$

$$s \longrightarrow -j-1$$

$$\sum_{j=-\infty}^{-1} (-1)^{-j-1} q^{\frac{3(-j-1)^2+5(-j-1)+2}{2}}$$

$$q^{\frac{3(j^2+2j+1)-5j-5+2}{2}} = q^{\frac{3j^2+j}{2}}$$

$$= \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{s(3s+1)}{2}}.$$