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MA 633 – Partition Theory – Lec. 13

- Recall that

$$\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)_\infty^{24}, \quad |q|<1.$$

Thm. 22 $\tau(n)$ is rarely odd. More precisely, $\tau(n)$ is odd iff n is an odd square. Moreover, the number of values of $n \le x$ for which $\tau(n)$ is odd equals $\left\lfloor \frac{1+\sqrt{x}}{2} \right\rfloor$.

Notation: The statement $f(q) \equiv g(q) \pmod{m}$ where $f(q) = \sum_{n=0}^{\infty} a(n) q^n$ & $g(q) = \sum_{n=0}^{\infty} b(n) q^n$, is equivalent to saying $a(n) \equiv b(n) \pmod{m}$ for every $n \ge 0$.

Proof: For any $j>0$, by binomial thm,

$$(1-q^j)^8 = \sum_{k=0}^{8} \binom{8}{k} (-q^j)^k$$

$$= 1 - 8q^j + 28q^{2j} - 56q^{3j} + 70q^{4j} - 56q^{5j} + 28q^{6j} - 8q^{7j} + q^{8j}$$

$$\equiv 1 + q^{8j} \pmod{2}$$

$$\equiv 1 - q^{8j} \pmod{2} \quad (\because 1 \equiv -1 \pmod{2}).$$

$$\Rightarrow \prod_{j=1}^{\infty}(1-q^j)^8 \equiv \prod_{j=1}^{\infty}(1-q^{8j}) \pmod 2$$

$$\Rightarrow (q;q)_\infty^8 \equiv (q^8;q^8)_\infty \pmod 2$$

Now

$$\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)_\infty^{24}$$

$$= q\{(q;q)_\infty^8\}^3$$

$$\equiv q(q^8;q^8)_\infty^3 \pmod 2$$

$$= \sum_{n=0}^{\infty} (-1)^n (2n+1) \cdot q^{8\left(\frac{n(n+1)}{2}\right)+1} \qquad \left[(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}}\right]$$

(by Jacobi's idty.)

$$= \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{4n^2+4n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{(2n+1)^2}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \tau(n) q^n \equiv \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{(2n+1)^2} \pmod 2.$$

Hence $\tau(n)$ is odd iff n is an odd square.

Also, $$\sum_{\substack{n \le x \\ \tau(n) \text{ is odd}}} 1 = \sum_{(2m-1)^2 \le x} 1$$

$$= \sum_{2m-1 \le \sqrt{x}} 1 = \sum_{m \le \frac{1+\sqrt{x}}{2}} 1 = \left\lfloor \frac{1+\sqrt{x}}{2} \right\rfloor .$$

$\boxtimes$

Thm. 23 For $n \ge 0$,
$\tau(7n), \tau(7n+3), \tau(7n+5), \tau(7n+6) \equiv 0 \pmod 7$.

Fact If p is a prime, then

$$p \mid \binom{p}{k} \text{ for any } 1 \le k \le p-1,$$

(can be proved using basic properties of $\nu_p(x)$: p-adic valuation of x (exponent of the highest power of p that divides x))

Proof: By binomial thm.,
$$(1-q^j)^7 \equiv 1 - q^{7j} \pmod 7 .$$

Hence $(q;q)_\infty^7 \equiv (q^7;q^7)_\infty \pmod 7$.

Now

$$\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)_\infty^{24}$$

$$= q(q;q)_\infty^3 \{(q;q)_\infty^7\}^3$$

$$\equiv q(q;q)_\infty^3 (q^7;q^7)_\infty^3 \pmod 7$$

The exponents in the power series expansion of $(q^7;q^7)_\infty^3$ are also multiples of 7.

Let us concentrate on $q(q;q)_\infty^3 \pmod 7$.

$$q(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}+1}.$$

For the first part ($\tau(7n) \equiv 0 \pmod 7$), we need to find out when $1+\frac{n(n+1)}{2} \equiv 0 \pmod 7$)

n	$1+\frac{n(n+1)}{2}$	$1+\frac{n(n+1)}{2} \pmod 7$
0	1	1
1	2	2
2	4	4
3	7	0
4	11	4
5	16	2
6	22	1

From the above table,

$1+\frac{n(n+1)}{2} \equiv 0 \pmod 7$ iff $n \equiv 3 \pmod 7$

But then $2n+1 = 7 \equiv 0 \pmod 7$ & hence the coeff. of $q^{1+\frac{n(n+1)}{2}}$ in $q(q;q)_\infty^3$ is congruent to $0 \pmod 7$ whenever $1+\frac{n(n+1)}{2} \equiv 0 \pmod 7$.

Hence $\tau(7n) \equiv 0 \pmod 7$

Since $1+\frac{n(n+1)}{2} \not\equiv 3, 5, 6 \pmod 7$,

we also get $\tau(7n+3), \tau(7n+5), \tau(7n+6) \equiv 0 \pmod 7$.

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