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MA 633 – Partition Theory – Lec. 14

Thm. 24 Let r, $0 \le r < 23$ denote any quadratic residue modulo 23.

(a is quadratic residue modulo n if $\exists x \ni x^2 \equiv a \pmod{n}$.)

Then for each $n \in \mathbb{N}$,

$$\tau(23n - r) \equiv 0 \pmod{23}.$$

Proof: $\displaystyle\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)_\infty^{24}$

$$= q(q;q)_\infty (q;q)_\infty^{23}$$

$$\equiv q(q;q)_\infty (q^{23};q^{23})_\infty \pmod{23}$$

Note that

$$q(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2} + 1}$$

Now $1 + \frac{n(3n+1)}{2} = (6n+1)^2 - \frac{23n(3n+1)}{2}$

$$\equiv (6n+1)^2 \pmod{23}$$

Thus $\tau(m)$ will be a multiple of 23 when m is not congruent to a square mod 23. $\longrightarrow$ $*$

In other words, this will happen if $m = 23n + l \neq k^2$, $0 \le l < 23$ for some integers n, l & k.

This implies that l must be a quadratic non-residue mod 23, because if l was a q.r. mod 23, then $\exists\, x \ni x^2 \equiv l \pmod{23}$

$$\Rightarrow m = 23n + l \equiv 23n + x^2 \pmod{23}$$
$$\equiv x^2 \pmod{23},$$

which would contradict $\circledast$.

From the hypotheses, r is q.r. mod 23.
Note that $23 \equiv -1 \pmod 4$.
& $\exists\, x \ni r \equiv x^2 \pmod{23}$.

$$\Leftrightarrow -r \equiv -x^2 \pmod{23}$$

and $-r = -x^2 + 23j \equiv -x^2 - j \pmod 4$

Now suppose that $-r$ is a q.r. (mod 23).
So $\exists\, y \ni -r \equiv y^2 \pmod{23}$

$$\Rightarrow -r = y^2 + 23t$$
$$\Rightarrow y^2 + 23t \equiv -x^2 - j \pmod 4$$
$$\Rightarrow y^2 + 23t = -x^2 - j + 4u$$
$$\Rightarrow x^2 + y^2 + j + 23t \equiv 4u.$$

Find the contradiction.

$\Rightarrow$ $-\gamma$ is q.n.r (mod 23).
(So $-\gamma$ is one of the l's)
$\Rightarrow$ $\tau(23n-\gamma) \equiv 0 \pmod{23}$.

Thm. 25 (Ramanujan)

$$p(5n+4) \equiv 0 \pmod 5 \quad \forall\, n \geq 0.$$

Proof:

$$q(q;q)_\infty^4 \cdot \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = q\frac{(q^5;q^5)_\infty}{(q;q)_\infty}$$

$$= (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)\, q^{m+1}$$

Note that

$$(q^5;q^5)_\infty \equiv (q;q)_\infty^5 \pmod 5$$

$$\Rightarrow \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv 1 \pmod 5$$

Hence

$$(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)\, q^{m+1} \equiv q(q;q)_\infty^4 \pmod 5$$

Then the congruence $p(5n+4) \equiv 0 \pmod 5$ will follow if we show that the coeff. of q^{5n+5} on the RHS is a mult. of 5.

Observe that

$$q(q;q)_\infty^4 = q(q;q)_\infty (q;q)_\infty^3$$

$$= q\left(\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}}\right)\left(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{\frac{k(k+1)}{2}}\right)$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} (2k+1) q^{1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}}$$

We now use the elementary identity

$$2(j+1)^2 + (2k+1)^2 = 8\left\{1 + \frac{j(3j+1)}{2} + \frac{k(k+1)}{2}\right\} - 10j^2 - 5$$

Hence $1 + \frac{j(3j+1)}{2} + \frac{k(k+1)}{2}$ will be a mult. of 5 iff $2(j+1)^2 + (2k+1)^2 \equiv 0 \pmod 5$

j	$(j+1)^2 \pmod 5$	$2(j+1)^2 \pmod 5$
0	1	2
1	4	3
2	4	3
3	1	2
4	0	0

k	$(2k+1)^2 \pmod 5$
0	1
1	4
2	0
3	4
4	1

So $2(j+1)^2 \equiv 0, 2, 3 \pmod 5$

& $(2k+1)^2 \equiv 0, 1, 4 \pmod 5$

Hence

$2(j+1)^2 + (2k+1)^2 \equiv 0 \pmod 5$ only when

$2(j+1)^2 \equiv 0 \pmod 5$ & $(2k+1)^2 \equiv 0 \pmod 5$

$$\Updownarrow$$

$2k+1 \equiv 0 \pmod 5$