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MA 633 – Partition Theory – Lec. 15

Let $p_m(n)$ denote the number of partitions of n into $\le m$ parts. Then

$$\sum_{n=0}^{\infty} p_m(n) q^n = \sum_{n_1 \ge n_2 \ge \cdots \ge n_m \ge 0} q^{n_1+n_2+\cdots+n_m}$$

Macmahon's idea: Introduce new variables $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ which will handle the inequalities satisfied by n_j while n_j themselves become free

Consider the sum

$$\sum_{n_1 \ge n_2 \ge \cdots n_m \ge 0} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \cdots \lambda_{m-1}^{n_{m-1}-n_m}$$

Define an operator $\underset{\ge}{\Omega}$ applied to any multiple Laurent series which annihilates terms with negative exponents of λ_i's and sets the λ_i's in the remaining to be 1,

$$\sum_{n=0}^{\infty} p_m(n) q^n = \underset{\ge}{\Omega} \sum_{n_1, n_2, \ldots n_m \ge 0} q^{n_1+n_2+\cdots n_m} \lambda_1^{n_1-n_2} \cdots \lambda_{m-1}^{n_{m-1}-n_m}.$$

$$= \underset{\ge}{\Omega} \left(\sum_{n_1=0}^{\infty} (q\lambda_1)^{n_1}\right) \sum_{n_2=0}^{\infty} \left(\frac{q\lambda_2}{\lambda_1}\right)^{n_2} \cdots \sum_{n_{m-1}=0}^{\infty} \left(\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)^{n_{m-1}} \sum_{n_m=0}^{\infty} \left(\frac{q}{\lambda_{m-1}}\right)^{n_m}$$

$$= \underset{\geq}{\Omega} \frac{1}{(1-q\lambda_1)\left(1-q\frac{\lambda_2}{\lambda_1}\right)\cdots\cdots\left(1-q\frac{\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

Lemma 26

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

Proof: LHS $= \underset{\geq}{\Omega} \left(\sum_{n=0}^{\infty} (\lambda x)^n\right) \sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)^m$

$$= \sum_{n \geq m \geq 0} x^n y^m$$

Let $k = n-m$. Then

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} x^{m+k} y^m$$

$$= \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{m=0}^{\infty} (xy)^m\right)$$

$$= \frac{1}{(1-x)(1-xy)}.$$

∎

Thm. 27 $$\sum_{n=0}^{\infty} p_m(n) q^n = \frac{1}{(q;q)_m}.$$

Proof: From the above discussion,

$$\sum_{n=0}^{\infty} p_m(n) q^n$$

$$= \underset{\geq}{\Omega} \frac{1}{(1-q\lambda_1)\left(1-\frac{q\lambda_2}{\lambda_1}\right)\cdots\left(1-\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

Use Lemma 26 with λ being λ_1 & $x = q$ & $y = q\lambda_2$ so that

$$= \underset{\geq}{\Omega} \frac{1}{(1-q)(1-q^2\lambda_2)\left(1-\frac{q\lambda_3}{\lambda_2}\right)\cdots\left(1-\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

Apply lemma 26 again with $x = q^2$, $y = q\lambda_3$ & $\lambda = \lambda_3$ so that

$$= \frac{1}{(1-q)} \underset{\geq}{\Omega} \frac{1}{(1-q^2)(1-q^3\lambda_3)\cdots\left(1-\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

$$= \frac{1}{(1-q)(1-q^2)\cdots(1-q^{m-2})} \underset{\geq}{\Omega} \frac{1}{(1-q^{m-1}\lambda_{m-1})\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

$$\overset{\text{(Lemma 26)}}{=} \frac{1}{(1-q)\cdots(1-q^m)} = \frac{1}{(q;q)_m}.$$

∎

Lemma 28 For $\alpha \in \mathbb{Z}$,

$$\underset{\lambda}{\Omega} \frac{\lambda^{-\alpha}}{(1-\lambda x)(1-\frac{y}{\lambda})} = \frac{x^{\alpha}}{(1-x)(1-xy)}$$

Proof:

$$LHS = \underset{\geq}{\Omega} \sum_{n=0}^{\infty} (\lambda x)^n \sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)^m \cdot \lambda^{-\alpha}$$

$$= \underset{\geq}{\Omega} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m \lambda^{n-m-\alpha}$$

$$= \sum_{n-m-\alpha=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m$$

Let $k = n-m-\alpha$ so that

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} x^{k+m+\alpha} y^m$$

$$= x^{\alpha} \left(\sum_{k=0}^{\infty} x^k \right) \sum_{m=0}^{\infty} (xy)^m$$

$$= \frac{x^{\alpha}}{(1-x)(1-xy)},$$

∎

Thm. 29 Let $\Delta(n)$ = number of incongruent triangles of perimeter n & integral sides. Then show that

$$\sum_{n=0}^{\infty} \Delta(n) q^n = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}$$

Note that the RHS is the g.f. of $a(n)$, where $a(n)$ = # of ptns. of n into 2's, 3's & 4's with at least one three.

$$\left(\because \frac{q^3}{1-q^3} = q^{1.3} + q^{2.3} + q^{3.3} + \cdots\cdots \right)$$

Proof: Let $n_1 \geq n_2 \geq n_3$ be the sides of the Δ so that $n_1+n_2+n_3 = n$,

& $n_2+n_3 \geq n_1+1$ (so as to not have non-degenerate Δ's)

Then

$$\sum_{n=0}^{\infty} \Delta(n) q^n = \sum_{\substack{n_1 \geq n_2 \geq n_3 \geq 0 \\ n_2+n_3 \geq n_1+1}} q^{n_1+n_2+n_3}$$

$$= \underset{\geq}{\Omega} \sum_{n_1,n_2,n_3=0}^{\infty} q^{n_1+n_2+n_3} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \lambda_3^{n_2+n_3-n_1-1}$$

$$= \underset{\geq}{\Omega} \lambda_3^{-1} \sum_{n_1=0}^{\infty} \left(\frac{q\lambda_1}{\lambda_3}\right)^{n_1} \sum_{n_2=0}^{\infty} \left(\frac{q\lambda_2\lambda_3}{\lambda_1}\right)^{n_2} \sum_{n_3=0}^{\infty} \left(\frac{q\lambda_3}{\lambda_2}\right)^{n_3}$$

$$= \underset{\geq}{\Omega} \frac{\lambda_3^{-1}}{\left(1-\frac{q\lambda_1}{\lambda_3}\right)\left(1-\frac{q\lambda_2\lambda_3}{\lambda_1}\right)\left(1-\frac{q\lambda_3}{\lambda_2}\right)}$$

Use Lemma 26 with $\lambda = \lambda_1$, $x = \frac{q}{\lambda_3}$, $y = q\lambda_2\lambda_3$

$$= \underset{\geq}{\Omega} \frac{\lambda_3^{-1}}{\left(1-\frac{q}{\lambda_3}\right)\left(1-q^2\lambda_2\right)\left(1-\frac{q\lambda_3}{\lambda_2}\right)}$$

Use Lemma 26 with $\lambda = \lambda_2$, $x = q^2$ and $y = q\lambda_3$.

$$= \underset{\geq}{\Omega} \frac{\lambda_3^{-1}}{\left(1-\frac{q}{\lambda_3}\right)\left(1-q^2\right)\left(1-q^3\lambda_3\right)}$$

$$= \frac{1}{1-q^2} \underset{\geq}{\Omega} \frac{\lambda_3^{-1}}{\left(1-q^3\lambda_3\right)\left(1-\frac{q}{\lambda_3}\right)}$$

$$= \frac{1}{(1-q^2)} \frac{q^3}{(1-q^3)(1-q^4)}.$$

Using $\displaystyle \Omega_{\geq} \frac{\lambda^{-\alpha}}{(1-\lambda x)(1-\frac{y}{\lambda})} = \frac{x^{\alpha}}{(1-x)(1-xy)}$

with $\alpha = 1$, $x = q^3$, $y = q$.