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MA 633 – Partition Theory – Lec. 16

Thm. 30 Let $d_m(n)$ denote the number of ptns. of n into exactly m distinct parts. Then

$$\sum_{n=0}^{\infty} d_m(n) q^n = \frac{q^{m(m+1)/2}}{(q;q)_m}$$

Proof: $n = n_1 + n_2 + \cdots + n_m$,

$n_1 \geq n_2 + 1$, $n_2 \geq n_3 + 1$, ……, $n_{m-1} \geq n_m + 1$, $n_m \geq 1$.

$$\sum_{n=0}^{\infty} d_m(n) q^n$$

$$= \underset{\geq}{\Omega} \sum_{n_1, n_2, \ldots, n_m \geq 0} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2-1} \lambda_2^{n_2-n_3-1} \cdots \lambda_{m-1}^{n_{m-1}-n_m-1} \times \lambda_m^{n_m-1}$$

$$= \underset{\geq}{\Omega} \frac{\lambda_1^{-1}\lambda_2^{-1}\cdots\lambda_m^{-1}}{(1-\lambda_1 q)\left(1-\frac{\lambda_2 q}{\lambda_1}\right)\cdots\left(1-\frac{\lambda_m q}{\lambda_{m-1}}\right)}$$

(by Lemma 26)

$$= \frac{q^{1+2+\cdots+m}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$= \frac{q^{m(m+1)/2}}{(q;q)_m}$$

Thm. 31 Let $p_m(j,n)$ (resp. $q_m(j,n)$) denote the number of partitions of n into at most m parts (resp. exactly m distinct parts) with the largest part j. Then

(i) $$\sum_{j,n=0}^{\infty} p_m(j,n)\, z^j q^n = \frac{1}{(zq;q)_m}$$

(ii) $$\sum_{j,n=0}^{\infty} q_m(j,n)\, z^j q^n = \frac{z^m q^{m(m+1)/2}}{(zq;q)_m}$$

Proof: (i)

$$n = n_1 + n_2 + \cdots + n_m$$

$$n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_m \geq 0, \quad n_1 = j.$$

$$\sum_{j,n=0}^{\infty} p_m(j,n)\, z^j q^n$$

$$= \underset{\geq}{\Omega} \sum_{n_1,n_2,\ldots,n_m=0}^{\infty} z^{n_1} q^{n_1+\cdots+n_m} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \cdots \lambda_{m-1}^{n_{m-1}-n_m}$$

$$= \underset{\geq}{\Omega} \frac{1}{(1-zq\lambda_1)\left(1-\frac{q\lambda_2}{\lambda_1}\right)\cdots\left(1-\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)\left(1-\frac{q}{\lambda_{m-1}}\right)}$$

$$= \frac{1}{(1-z^{\circ}q)(1-zq)\cdots(1-zq^{m})}$$ $\boxtimes$

Notation: $[z^j] \sum_{n=0}^{\infty} a_n z^n = a_j$.

Observation: $\ast$

$$\sum_{h=0}^{N} a_h = \sum_{h=0}^{N} [z^h] \sum_{n=0}^{\infty} a_n z^n$$

$$= [z^N](1+z+z^2+\cdots)(a_0+a_1 z+a_2 z^2+\cdots)$$

[since

$$(1+z+z^2+\cdots)(a_0+a_1 z+\cdots)$$
$$= a_0+(a_1+a_0)z+(a_2+a_1+a_0)z^2+\cdots]$$

$$= [z^N] \frac{\sum_{n=0}^{\infty} a_n z^n}{1-z}.$$

Thm. 32 Suppose $p(N, M, n)$ denote the number of partitions of n into $\leq M$ parts with each part $\leq N$.

Then

$$\sum_{n=0}^{\infty} p(N,M,n) q^n = \begin{bmatrix} N+M \\ M \end{bmatrix}_q = \frac{(q)_{N+M}}{(q)_N (q)_M}$$

Proof: $$\sum_{n=0}^{\infty} p(N,M,n)q^n = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{N} p_M(j,n)\right)q^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{N} [z^j]\sum_{k=0}^{\infty} p_M(k,n)z^k\right)q^n.$$

$$= \sum_{j=0}^{N}[z^j]\sum_{n,k=0}^{\infty} p_M(k,n)z^k q^n$$

$$= \sum_{j=0}^{N}[z^j]\frac{1}{(1-zq)(1-zq^2)\cdots(1-zq^M)}$$

We now prove a lemma,

Lemma 33 $$\frac{1}{(z)_N} = \sum_{j=0}^{\infty}\begin{bmatrix}N+j-1\\ j\end{bmatrix}z^j$$

Proof: $$\frac{1}{(z)_N} = \frac{(zq^N)_\infty}{(z)_\infty}$$

$$= \sum_{j=0}^{\infty}\frac{(q^N)_j}{(q)_j}z^j$$

$$\left[\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty}\right]$$

$$= \sum_{j=0}^{\infty} \frac{(q)_{N-1}(q^N)_j}{(q)_j (q)_{N-1}} z^j$$

$$= \sum_{j=0}^{\infty} \frac{(q)_{N+j-1}}{(q)_j (q)_{N-1}} z^j$$

$$= \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j$$

$$\begin{bmatrix} N+M \\ M \end{bmatrix} = \begin{bmatrix} N+M \\ N \end{bmatrix}$$

$\|$ Ex. 2 (long time back)

$\|$

$$\sum_{j=0}^{N} \begin{bmatrix} M+j-1 \\ j \end{bmatrix} q^j$$

We now continue with the proof of Thm. 32.

Note that

$$\sum_{j=0}^{N} [z^j] \sum_{k=0}^{\infty} \begin{bmatrix} M+k-1 \\ k \end{bmatrix} (zq)^k$$

(Alt. proof)

$$\sum_{n=0}^{\infty} p(N,M,n)\, q^n = \sum_{j=0}^{N} [z^j] \frac{1}{(1-zq)(1-zq^2)\cdots(1-zq^M)}$$

$\overset{\star}{=}$ Observation

$$[z^N] \frac{1}{(1-z)(1-zq)\cdots(1-zq^M)} = [z^N] \frac{1}{(z;q)_{M+1}}$$

$$= [z^N] \sum_{j=0}^{\infty} \begin{bmatrix} M+j \\ j \end{bmatrix} z^j = \begin{bmatrix} N+M \\ N \end{bmatrix}.$$