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MA 633 – Partition Theory – Lec. 17

Thm. 32 Suppose $p(N,M,n)$ denote the number of partitions of n into $\leq M$ par with each part $\leq N$.
Then
$$\sum_{n=0}^{\infty} p(N,M,n)\, q^n = \begin{bmatrix} N+M \\ M \end{bmatrix}_q = \frac{(q)_{N+M}}{(q)_N (q)_M} \qquad \text{—— } (*)$$

Alternative proof:

Note that $p(N,M,n)=0$ if $n > MN$.

& $p(N,M,MN) = 1$.

Hence the generating fn.
$$G(N,M;q) = \sum_{n=0}^{\infty} p(N,M,n)\, q^n$$

is a polynomial in q of deg. MN,

Proof: Let $g(N,M;q)$ denote the RHS of $(*)$.

Note that $g(N,0;q) = \dfrac{(q)_{N+0}}{(q)_N (q)_0} = 1$,

$$\text{——— } (a)$$

Also, $g(0, M; q) = \dfrac{(q)_{0+M}}{(q)_0 (q)_M} = 1$ —— (b)

&

$g(N, M; q) - g(N, M-1; q)$

$$= \frac{(q)_{N+M}}{(q)_N (q)_M} - \frac{(q)_{N+M-1}}{(q)_N (q)_{M-1}}$$

$$= \frac{(q)_{N+M-1}}{(q)_N (q)_M} \left\{ (1 - q^{N+M}) - (1 - q^M) \right\}$$

$$= \frac{(q)_{N+M-1}}{(q)_N (q)_M} q^M (1 - q^N)$$

$\Rightarrow$

$$g(N, M; q) - g(N, M-1; q) = q^M g(N-1, M; q) \quad \text{—— (c)}$$

By principle of double induction on N & M, it can be shown that $g(N, M; q)$ is the unique fn. satisfying (a), (b) & (c).

<u>Goal</u>: To show that $G(N, M; q)$ also satisfies (a), (b) & (c).

Now $G(0, M; q) = \sum_{n=0}^{\infty} p(0, M, n) q^n$

$= 1$ ——— (b)

$$p(N, 0, n) = p(0, M, n) = \begin{cases} 1, & \text{if } N = M = n = 0 \\ 0, & \text{else} \end{cases}$$

because the empty partition of 0 is the only partition in which no part is positive and also the only partition in which the number of parts is non-positive,

Also $G(N, 0; q) = 1$ ——— (a)

We next show that

$$G(N, M; q) - G(N, M-1; q) = q^M G(N-1, M; q).$$

Note $p(N, M, n) - p(N, M-1, n)$

= number of ptns. of n into exactly M parts each $\leq N$. ——— (**)

We show that

$p(N, M, n) - p(N, M-1, n) = p(N-1, M, n-M)$.

This is done by bijectively mapping a partition in (**) with a partition enumerated by $p(N-1, M, n-M)$ (by deleting one node from each part of a partition of (**)).

Hence $\sum_{n=0}^{\infty} p(N,M,n)\, q^n - \sum_{n=0}^{\infty} p(N,M-1,n)\, q^n$

$$= \sum_{n=0}^{\infty} p(N-1,M,n-M)\, q^n$$

$$= q^M \sum_{n=0}^{\infty} p(N-1,M,n-M)\, q^{n-M}.$$

or, in other words,

$$\left. \begin{aligned} &G(N,M;q) - G(N,M-1;q) \\ &= q^M\, G(N-1,M;q). \end{aligned} \right\} \text{ so (C) is also satisfied}$$

By uniqueness,

$$G(N,M;q) = g(N,M;q) = \begin{bmatrix} N+M \\ M \end{bmatrix}.$$

$\boxtimes$

<u>Notation</u>: (Basic hypergeometric series)

$${}_2\phi_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n, \quad \left(\begin{array}{l}\text{abs. conv. for} \\ |z|<1 \\ |q|<1\end{array}\right)$$

where $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

This is a q-analogue of Gaussian hypergeometric series:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \left(\begin{array}{l}\text{Abs. conv. for} \\ |z|<1\end{array}\right)$$

where $(a)_n = a(a+1)\cdots(a+n-1)$

$${}_r\phi_{r-1}\left(a_1, a_2, \ldots, a_r; c_1, c_2, \ldots c_{r-1}; z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{(c_1)_n \ldots (c_{r-1})_n (q)_n} z^n$$

<u>Thm.34</u> (Heine's transformation)

For $|q|<1$, $|z|<1$, $|b|<1$,

$${}_2\phi_1(a, b; c; z) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1\left(\frac{c}{b}, z; az; b\right)$$

<u>Proof</u>: $${}_2\phi_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \frac{(cq^n)_\infty}{(bq^n)_\infty} z^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \left(\sum_{m=0}^{\infty} \frac{(c/b)_m}{(q)_m} (bq^n)^m \right) z^n$$

$$\left(\text{since } \sum_{n=0}^{\infty} \frac{(A)_n}{(q)_n} Z^n = \frac{(AZ)_\infty}{(Z)_\infty}. \right.$$

Now let $Z = bq^n$

& $AZ = cq^n \Longrightarrow A = \dfrac{cq^n}{bq^n} = \dfrac{c}{b}$, $\Big)$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m}{(q)_m} b^m \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} (zq^m)^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m}{(q)_m} b^m \cdot \frac{(azq^m)_\infty}{(zq^m)_\infty}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m}{(q)_m} b^m \cdot \frac{(azq^m)_\infty}{(zq^m)_\infty} \frac{(az)_m}{(az)_m} \frac{(z)_m}{(z)_m}$$

$$= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (z)_m}{(az)_m (q)_m} b^m$$

$$= \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty} {}_2\varphi_1\left(c/b, z; az; b \right)$$

$\square$