11/9/21

MA 633 – Partition Theory – Lec. 18

Last time: Heine's transformation

$${}_2\varphi_1(a,b;c;z) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty}\, {}_2\varphi_1\left(\frac{c}{b}, z; az; b\right),$$

Thm. 3.5: (q-analogue of Gauss' thm.)

$$\underbrace{\left({}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \mathrm{Re}(c-a-b)>0\right)}_{\text{Gauss' thm.}}$$

For $|c|<|ab|$ & $|q|<1$,

$${}_2\varphi_1\left(a,b;c;\frac{c}{ab}\right) = \frac{(c/a)_\infty (c/b)_\infty}{(c/ab)_\infty (c)_\infty}$$

Proof: ${}_2\varphi_1\left(a,b;c;\frac{c}{ab}\right)$

$$\overset{\text{(Heine)}}{=} \frac{(b)_\infty (c/b)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty}\, {}_2\varphi_1\left(\frac{c}{b}, \frac{c}{ab}; \frac{c}{b}; b\right)$$

$$= \frac{(b)_\infty (c/b)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty} \sum_{n=0}^{\infty} \frac{\cancel{(c/b)_n}\,(c/ab)_n}{\cancel{(c/b)_n}\,(q)_n}\, b^n$$

(q-binomial)

$$= \frac{\cancel{(b)_\infty}(c/b)_\infty}{(c)_\infty\left(\frac{c}{ab}\right)_\infty} \cdot \frac{(c/a)_\infty}{\cancel{(b)_\infty}}.$$

we require $|b|<1$ but it can be relaxed by analytic continuation

$$= \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty}.$$

Thm. 36 For $|q| < \min(1, |b|)$,

$$ {}_2\varphi_1\left(a, b; \frac{aq}{b}; \frac{-q}{b}\right) = \frac{(aq;q^2)_\infty (-q;q)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty \left(\frac{-q}{b};q\right)_\infty}$$

Proof: (This theorem is due to W.N. Bailey).

$$ {}_2\varphi_1\left(a, b; \frac{aq}{b}; \frac{-q}{b}\right) = {}_2\varphi_1\left(b, a; \frac{aq}{b}; \frac{-q}{b}\right)$$

Heine: For $|z|<1$, $|B|<1$, $|q|<1$:

$$ {}_2\varphi_1(A, B; C; Z) = \frac{(B)_\infty (AZ)_\infty}{(C)_\infty (Z)_\infty} {}_2\varphi_1\left(\frac{C}{B}, Z; AZ; B\right)$$

(Heine)

$$= \frac{(a)_\infty (-q)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{-q}{b}\right)_\infty} {}_2\varphi_1\left(\frac{q}{b}, \frac{-q}{b}; -q; a\right)$$

(for $\left|\frac{-q}{b}\right| < 1$ & $|a| < 1$)

$$= \frac{(a)_\infty(-q)_\infty}{\left(\frac{aq}{b}\right)_\infty\left(\frac{-q}{b}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b}\right)_n\left(\frac{-q}{b}\right)_n}{(-q)_n(q)_n} a^n$$

$$= \frac{(a)_\infty(-q)_\infty}{\left(\frac{aq}{b}\right)_\infty\left(\frac{-q}{b}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q^2}{b^2};q^2\right)_n}{(q^2;q^2)_n} a^n$$

(q-binomial)

$$= \frac{(a)_\infty(-q)_\infty}{\left(\frac{aq}{b}\right)_\infty\left(\frac{-q}{b}\right)_\infty} \cdot \frac{\left(\frac{aq^2}{b^2};q^2\right)_\infty}{(a;q^2)_\infty}$$

$$= \frac{(aq;q^2)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(\frac{-q}{b};q\right)_\infty}$$

(using $(a;q)_\infty = (a;q^2)_\infty(aq;q^2)_\infty$)

Cor. 37

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} q^{n(n+1)/2} = (aq;q^2)_\infty(-q;q)_\infty.$$

Proof: By Bailey's Thm., for $|q| < \min(1, |b|)$,

$${}_2\varphi_1\left(a, b; \frac{aq}{b}; \frac{-q}{b}\right) = \frac{(aq;q^2)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(\frac{-q}{b};q\right)_\infty}$$

Let $b \to \infty$.

$$LHS = \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\left(\frac{aq}{b}\right)_n (q)_n} \left(-\frac{q}{b}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (-q)^n}{(q)_n} \lim_{b\to\infty} (b)_n \left(\frac{1}{b}\right)^n$$

Note that

$$\lim_{b\to\infty} (b)_n \left(\frac{1}{b}\right)^n = \lim_{b\to\infty} \frac{(1-b)}{b} \frac{(1-bq)}{b} \cdots \frac{(1-bq^{n-1})}{b}$$

$$= \lim_{b\to\infty} \left(\frac{1}{b}-1\right)\left(\frac{1}{b}-q\right) \cdots \left(\frac{1}{b}-q^{n-1}\right)$$

$$= (-1)^n q^{1+2+\cdots+n-1}$$

$$= (-1)^n q^{n(n-1)/2}$$

Hence

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} q^{n(n+1)/2} = (aq;q^2)_\infty (-q;q)_\infty .$$

Another proof of Sylvester's refinement of Euler's theorem (V. Ramamani & K. Venkatachaliengar)

Goal: $A_k(n) = B_k(n)$

Proof: $\displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)\, a^k q^n$

$$= \prod_{j=1}^{\infty}\left(1 + aq^{2j-1} + aq^{2(2j-1)} + aq^{3(2j-1)} + \cdots\right)$$

$$= \prod_{j=1}^{\infty}\left[1 + aq^{2j-1}\left(1 + q^{2j-1} + q^{2(2j-1)} + \cdots\right)\right]$$

$$= \prod_{j=1}^{\infty}\left(1 + \frac{aq^{2j-1}}{1-q^{2j-1}}\right)$$

$$= \prod_{j=1}^{\infty}\left(\frac{1-(1-a)q^{2j-1}}{1-q^{2j-1}}\right)$$

$$= \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty}.$$

If we now directly use the partitions λ enumerated by $B_k(n)$ to calculate

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)\, a^k q^n,$$

it is quite tough.

So instead, we examine the conjugates of the partitions enumerated by $B_k(n)$, denoted by λ'.

We need consider 2 separate cases of such partitions: Let λ

<u>Case 1</u>: 1 is a part of λ

Then λ' is described as follows:

- Has unique largest part
- All parts less than the largest part appear as parts (since λ is a ptn. into distinct parts)
- Exactly $k-1$ parts appear more than once.