Another proof of Sylvester's refinement of Euler's theorem (V. Ramamani & K. Venkatachaliengar)

Goal: $A_k(n) = B_k(n)$

Proof: $\displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)\, a^k q^n$

$$= \prod_{j=1}^{\infty}\left(1 + aq^{2j-1} + aq^{2(2j-1)} + aq^{3(2j-1)} + \cdots\right)$$

$$= \prod_{j=1}^{\infty}\left[1 + aq^{2j-1}\left(1 + q^{2j-1} + q^{2(2j-1)} + \cdots\right)\right]$$

$$= \prod_{j=1}^{\infty}\left(1 + \frac{aq^{2j-1}}{1-q^{2j-1}}\right)$$

$$= \prod_{j=1}^{\infty}\left(\frac{1-(1-a)q^{2j-1}}{1-q^{2j-1}}\right)$$

$$= \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty} = ((1-a)q;q^2)_\infty(-q;q)_\infty \quad \longrightarrow \star_1$$

If we now directly use the partitions λ enumerated by $B_k(n)$ to calculate

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)\, a^k q^n,$$ it is quite tough.

So instead, we examine the conjugates of the partitions enumerated by $B_k(n)$, denoted by λ'.

We need consider 2 separate cases of such partitions: Let λ

Case 1: 1 is a part of λ

Then λ' is described as follows:

- Has unique largest part
- All parts less than the largest part appear as parts (since λ is a ptn. into distinct parts)
- Exactly $k-1$ parts appear more than once.

(when a seq. is disturbed, then some part in the conjugate part repeats :)

Case 2: 1 is not a part of λ

Then λ' is described as follows:

- The largest part repeats
- All parts less than the largest part appear as parts (since λ is a ptn. into distinct parts)
- Exactly $\underbrace{k \text{ parts}}_{=\,(k-1)\text{ from previous case } + \text{ largest part}}$ appear more than once.

15/9/21

MA 633 – Partition Theory – Lec. 19

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)\, a^k q^n$$

unique largest part

doesn't begin with 1 since all parts less than the largest part appear as parts

$$= 1 + \sum_{N=1}^{\infty} a q^N \prod_{j=1}^{N-1} \left(q^j + a q^{2j} + a q^{3j} + \cdots\right)$$

$$+ \sum_{N=1}^{\infty} \underbrace{\left(a q^{2N} + a q^{3N} + \cdots\right)}_{\text{repeating largest part}} \prod_{j=1}^{N-1} \left(q^j + a q^{2j} + a q^{3j} + \cdots\right)$$

no 'a' because only repeating parts contribute to the power of a.

repeating largest part

$$= 1 + \sum_{N=1}^{\infty} \left(a q^N + a q^{2N} + a q^{3N} + \cdots\right) \prod_{j=1}^{N-1} \left(q^j + a q^{2j} + a q^{3j} + \cdots\right)$$

$$= 1 + \sum_{N=1}^{\infty} \frac{a q^N}{1-q^N} \prod_{j=1}^{N-1} q^j \left(1 + a q^j \left(1 + q^j + \cdots\right)\right)$$

$$= 1 + \sum_{N=1}^{\infty} \frac{a q^N}{1-q^N} \prod_{j=1}^{N-1} q^j \left(1 + \frac{a q^j}{1-q^j}\right)$$

$$= 1 + \sum_{N=1}^{\infty} \frac{a q^{\frac{N(N+1)}{2}}}{1-q^N} \prod_{j=1}^{N-1} \left(\frac{1-(1-a)q^j}{1-q^j}\right)$$

$$= 1 + \sum_{N=1}^{\infty} \frac{a q^{\frac{N(N+1)}{2}}}{1-q^N} \frac{((1-a)q;q)_{N-1}}{(q;q)_{N-1}}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{(1-a;q)_N}{(q;q)_N} q^{\frac{N(N+1)}{2}} \quad \left(\because a = 1-(1-a)\right)$$

$$= \sum_{N=0}^{\infty} \frac{(1-a;q)_N}{(q;q)_N} q^{N(N+1)/2}$$

$$= \left((1-a)q;q^2\right)_\infty (-q;q)_\infty \quad \left(\text{from Cor. 37}\right)$$

$$— \bigstar_2$$

From $\bigstar_1$ & $\bigstar_2$, we see that

$$A_k(n) = B_k(n).$$

Thm. 38 (Fine's refinement of Euler's theorem)

The number of partitions of n into distinct parts with the largest part being k is equal to the partitions of n into odd parts such that $2k+1$ = largest part + twice the number of parts.

$A_k(n)$

$B_k(n)$

Proof: Goal:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)\, t^k q^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)\, t^k q^n.$$

Now

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k(n)\, t^k q^n = 1 + \sum_{j=1}^{\infty} \underbrace{(-q)_{j-1}}_{\text{distinct parts}}\; t^j \underbrace{q^j}_{\text{largest part}}.$$

unique largest part

To get a representation for $\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B_k(n)\, t^k q^n$:

We claim that the required representation is

$$1 + \sum_{j=1}^{\infty} \frac{t^j q^{2j-1}}{(tq;q^2)_j},$$

q^{2j-1} ← largest part

Reason:

(i) the number of parts $= k+1-j$, because

largest part + twice the number of parts

$$= 2j-1 + 2(k+1-j)$$

$$= 2j-1+2k+2-2j$$

$$= 2k+1$$

(ii) $$\frac{1}{(tq;q^2)_j} = \frac{1}{(1-tq)(1-tq^3)\cdots(1-tq^{2j-1})}$$

$$= \sum_{m_1, m_2, \ldots m_j = 0}^{\infty} t^{m_1+m_2+\cdots+m_j} \; q^{1\cdot m_1 + 3m_2 + 5m_3 + \cdots + (2j-1)m_j}$$

Along with the largest part represented by the numerator of $\displaystyle\sum_{j=1}^{\infty} \frac{t^j q^{2j-1}}{(tq;q^2)_j}$,

this implies that

$m_1+m_2+\cdots+m_j+1$ = number of parts $= k+1-j$

$$\Rightarrow \underbrace{m_1+m_2+\cdots+m_j}_{\text{power of } t \text{ in } \frac{q^{2j-1}}{(tq;q^2)_j}} = k-j$$

(iii) But we want the power of t to be k & not $k-j$, hence we need to multiply $\dfrac{q^{2j-1}}{(tq;q^2)_j}$ with t^j.

Hence this shows that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_k(n) t^k q^n = 1 + \sum_{j=1}^{\infty} \frac{t^j q^{2j-1}}{(tq;q^2)_j},$$