16/9/21

MA 633 - Partition Theory - Lec. 20

Goal: $$\sum_{j=1}^{\infty} \frac{t^j q^{2j-1}}{(tq;q^2)_j} = \sum_{j=1}^{\infty} (-q)_{j-1} t^j q^j \quad \text{---} (\S)$$

Proof of $(\S)$:

$$\sum_{j=1}^{\infty} (-q)_{j-1} t^j q^j = tq \sum_{j=0}^{\infty} (-q)_j t^j q^j$$

$$= tq \sum_{j=0}^{\infty} \frac{(q^2;q^2)_j}{(q)_j} (tq)^j$$

$$= tq (q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{(tq)^j}{(q)_j} \frac{1}{(q^{2j+2};q^2)_\infty}$$

$$= tq (q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{(tq)^j}{(q)_j} \sum_{m=0}^{\infty} \frac{q^{(2j+2)m}}{(q^2;q^2)_m} \leftarrow \text{using } \left(\sum \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty} \right)$$

$$= tq (q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \sum_{j=0}^{\infty} \frac{(tq^{2m+1})^j}{(q;q)_j}$$

$$= tq (q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \frac{1}{(tq^{2m+1};q)_\infty}$$

$$= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m}(tq)_{2m}}{(q^2;q^2)_m}$$

$$= \frac{tq(q^2;q^2)_\infty}{(tq)_\infty} \sum_{m=0}^{\infty} \frac{(tq;q^2)_m (tq^2;q^2)_m}{(q^2;q^2)_m} q^{2m}$$

Using Heine's transformation,

$${}_2\phi_1(a,b;c;z) = \frac{(b)_\infty (az)_\infty}{(c)_\infty (z)_\infty}\, {}_2\phi_1\left(\frac{c}{b}, z; az; b\right),$$

we have

$$= \frac{tq(q^2;\cancel{q^2})_\infty}{(tq)_\infty} \cdot \frac{(tq^2;q^2)_\infty (tq^3;q^2)_\infty}{(q^2;\cancel{q^2})_\infty} \sum_{m=0}^{\infty} \frac{(q^2;\cancel{q^2})_m (tq^2)^m}{(tq^3;q^2)_m (q^2;\cancel{q^2})_m}$$

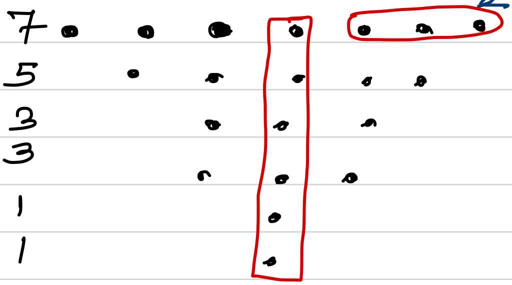
$$= \frac{tq(tq^2;q)_\infty}{(tq)_\infty} \sum_{m=0}^{\infty} \frac{(tq^2)^m}{(tq^3;q^2)_m}$$

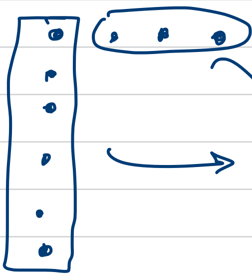
$$= tq \sum_{m=0}^{\infty} \frac{(tq^2)^m}{(tq;q^2)_{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m+1} q^{2m+1}}{(tq;q^2)_{m+1}} \overset{(m=j-1)}{=} \sum_{j=1}^{\infty} \frac{t^j q^{2j-1}}{(tq;q^2)_j}.$$

Remark: Fine's refinement of Euler's theorem can be obtained from Sylvester's bijection for his refinement of Euler's theorem.

largest part $\rightarrow \dfrac{\ell(\pi)-1}{2}$





k

$$\underbrace{\frac{\ell(\pi)-1}{2} + \text{number of parts}} = \text{largest part in the other class}$$

(ptns. into distinct parts with k seq. of consec. integers)

$$\ell(\pi) - 1 + 2 \cdot \text{number of parts} = 2k$$

$$\underbrace{\ell(\pi) + 2 \cdot \text{number of parts} = 2k+1}$$

Condition in Fine's refinement.

Thm. 39 Prove that

$$\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} (q;q^2)_n, & \text{if } m=2n \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Proof:

Consider

$$\sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} \right) \frac{z^m}{(q)_m}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j z^m}{(q)_j (q)_{m-j}}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j z^m}{(q)_j (q)_{m-j}}$$

$$\left(\because \frac{1}{(q)_{m-j}} = 0, \text{ when } j>m \right)$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(q)_j} \sum_{m=0}^{\infty} \frac{z^m}{(q)_{m-j}}$$

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}$$

$$(a)_{-1} = \frac{(a)_\infty}{\left(\frac{a}{q}\right)_\infty}$$

$$(q)_{-1} = \frac{(q)_\infty}{(1)_\infty} = \infty$$

$$\Rightarrow \frac{1}{(q)_{-1}} = 0$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(q)_j} \sum_{m=j}^{\infty} \frac{z^m}{(q)_{m-j}}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(q)_j} \sum_{m=0}^{\infty} \frac{z^m}{(q)_m}$$

$$= \sum_{j=0}^{\infty} \frac{(-z)^j}{(q)_j} \cdot \frac{1}{(z)_\infty}$$

$$= \frac{1}{(-z)_\infty} \cdot \frac{1}{(z)_\infty}$$

$$= \frac{1}{(z^2;q^2)_\infty}$$

$$= \sum_{m=0}^{\infty} \frac{z^{2m}}{(q^2;q^2)_m}$$

$$\Longrightarrow \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} \right) \frac{z^m}{(q)_m} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(q^2;q^2)_n}$$

Comparing the coeff. of z^m on both sides,

$$\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} \frac{1}{(q)_m} = \begin{cases} \frac{1}{(q^2;q^2)_n}, & \text{if } m=2n \\ 0, & \text{else} \end{cases}$$

Multiply both sides by $(q;q)_m$

$$\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} \frac{(q;q)_{2n}}{(q^2;q^2)_n}, & \text{if } m=2n \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} (q;q^2)_n, & \text{if } m=2n \\ 0, & \text{else} \end{cases}$$