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MA 633 – Partition Theory – Lec. 21

- Rank and crank of a partition

- In 1919, Ramanujan found

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

(1944) Dyson found a partition statistic which distributes partitions of $5n+4$ & $7n+5$ into 5 & 7 equinumerous classes respectively.

This statistic is called the rank of a partition, and is defined by

largest part − number of parts.

$\Rightarrow$ $\text{rank}(\pi) = l(\pi) - \#\text{ parts}$.

(π is a partition)

Consider 5 partitions of 4.

ptn. of 4	rank	rank (mod 5)
4	$4-1=3$	3
3+1	$3-2=1$	1
2+2	$2-2=0$	0
2+1+1	$2-3=-1$	4
1+1+1+1	$1-4=-3$	2

Consider 7 partitions of 5

	rank	rank (mod 7)
5	4	4
4+1	2	2
3+2	1	1
3+1+1	0	0
2+2+1	−1	6
2+1+1+1	−2	5
1+1+1+1+1	−4	3

We see that every representative of a residue class mod 7 appears once & only once.

Let $N(r, m, n)$ denote the number of partitions of n with rank $\equiv r \pmod{m}$. Then Dyson conjectured that

$$N(0,5,5n+4) = N(1,5,5n+4) = N(2,5,5n+4) = N(3,5,5n+4) = N(4,5,5n+4).$$

& similarly for 7.

- Dyson's conjecture was proved by Atkin & Swinnerton-Dyer.

But rank didn't do a similar thing for partitions of $11n+6$.

Consider, for example the 11 partitions of 6

	rank	rank (mod 11)
6	5	5
5+1	3	3
4+2	2	2
4+1+1	1	1
3+3	1	1
3+2+1	0	0
3+1+1+1	−1	10
2+2+2	−1	10
2+2+1+1	−2	9
2+1+1+1+1	−3	8
1+1+1+1+1+1	−5	6

repetit-ions

Dyson hypothesized the existence of another partition statistic, called crank, which will do the job.

- 44 years later, the crank was found by George Andrews & Frank Garvan!