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MA 633 – Partition Theory – Quiz 2

(1) Prove

$$1+\sum_{n=1}^{\infty} q^n(-q^{n+1})_\infty = (-q;q)_\infty .$$

n is the smallest part

g.f. for ptns. into distinct parts

(2) For every $N \in \mathbb{N}\cup\{0\}$,

$$(-q;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} q^{j(j+1)/2}$$

$\begin{bmatrix} N \\ j \end{bmatrix} = \begin{bmatrix} N-j+j \\ j \end{bmatrix}$ ← g.f. for ptns. of some integer into at most j parts, each $\le N-j$.

$j=7 \quad N=13$

at most 7 parts each ≤ 6

(3) $$(-q;q)_\infty = 1+\sum_{n=1}^{\infty} q^n (-q;q)_{n-1}$$

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Garvan in his PhD thesis obtained the crank for vector partitions.

π : partition of n
$\#(\pi)$: number of parts of π.

$V = \{(\pi_1, \pi_2, \pi_3) : \pi_1$ is a partition into distinct parts, π_2 & π_3 are ordinary partitions$\}$.

Let $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V$ be a vector partition of n, $|\pi_1| + |\pi_2| + |\pi_3| = n$,

Let $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$ (weight)
$r(\vec{\pi}) = \#\pi_2 - \#\pi_3$.

(Crank for vector partitions:)

$\vec{\pi} = (5+3+2, 2+2+1, 2+1+1)$
$\omega(\pi) = (-1)^{\#(\pi_1)} = (-1)^3 = -1$

$r(\pi) = 3 - 3 = 0$,

Let $N_V(m,n)$ be the number of vector partitions of n with crank m.
Then
$$N_V(m,n) = \sum_{\substack{\vec{\pi} \in V \\ |\pi_1| + |\pi_2| + |\pi_3| = n \\ r(\vec{\pi}) = m}} (-1)^{\#(\pi_1)}.$$

Thm. $$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty},$$

CRANK FOR ORDINARY PARTITIONS

Let π be a partition of n.

$\ell(\pi)$: largest part of π

$\omega(\pi)$: number of 1's in π.

$\mu(\pi)$: number of parts of π greater than $\omega(\pi)$.

Then crank $c(\pi)$

$$:= \begin{cases} \ell(\pi), & \text{if } \omega(\pi) = 0 \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0 \end{cases}$$

Example

ptns. of 6	$\ell(\pi)$	$\omega(\pi)$	$\mu(\pi)$	$c(\pi)$	$c(\pi) \pmod{11}$
6	6	0	—	6	6
5+1	5	1	1	0	0
4+2	4	0	—	4	4
4+1+1	4	2	1	−1	10
3+3	3	0	—	3	3
3+2+1	3	1	2	1	1
3+1+1+1	3	3	0	−3	8
2+2+2	2	0	—	2	2
2+2+1+1	2	2	0	−2	9
2+1+1+1+1	2	4	0	−4	7
1+1+1+1+1+1	1	6	0	−6	5

- Cranks: really the final problem by Bruce C. Berndt & his co-authors (2007).

- Crank not only distributes partitions of $11n+6$ into 11 equinumerous classes, but also those of $5n+4$ & $7n+5$.

Generating functions of ranks & cranks

Thm. 40 Let $N(m,n)$ denote the number of partitions of n with rank m. Then

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)\, z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}.$$

Remark: ① m really runs only for $-(n-1)$ to $(n-1)$.

② $$1+\sum_{n=1}^{\infty}\underbrace{\left(\sum_{m=-(n-1)}^{n-1} N(m,n)\right)}_{=\,p(n)} q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n^2}$$