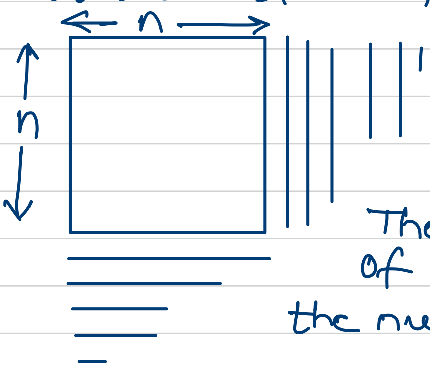
23/9/21

MA 633 – Partition Theory – Lec. 23

Thm. 40 Let $N(m,n)$ denote the number of partitions of n with rank m. Then

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)\, z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}.$$

Proof: Let n in the summand of RHS denote the side of the Durfee square of a partition of some integer.



The Durfee square contributes to q^{n^2} — ①

The partition to the right of D.S. is the one in which the number of parts is $\leq n$. — ②

So consider $\frac{1}{(zq)_n}$ from the summand. It generates the partitions in ② with z keeping track of the number of parts.

By conjugation, $\frac{1}{(zq)_n}$ counts the number of partitions with the largest part $\leq n$ & & with z keeping track of the largest part. — ③

Also $\dfrac{1}{(z^{-1}q)_n}$ generates the partition below the D.S. with z^{-1} keeping track of its number of parts.

Hence power of z in the whole summand on RHS keeps track

largest part in 2 − number of parts in 3

= (largest part in 2 + n)

− (number of parts in 3 + n)

= largest part − number of parts

= rank.

This completes the proof. ∎

Thm. 41 Let $M(m,n)$ denote the number of partitions of n with crank m. Then

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)\, z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}.$$

Proof:

$$\frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} = \frac{(1-q)}{(zq)_\infty}\,\frac{(q^2)_\infty}{(z^{-1}q)_\infty}$$

az

z

$$= \frac{1-q}{(zq)_\infty} \sum_{j=0}^{\infty} \frac{(zq)_j}{(q)_j} (z^{-1}q)^j \quad \text{(by } q\text{-binomial theorem)}$$

$$= \frac{1-q}{(zq)_\infty} + \frac{1-q}{(zq)_\infty} \sum_{j=1}^{\infty} \frac{(zq)_j}{(q)_j} (z^{-1}q)^j$$

$$= \frac{1-q}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{(z^{-1}q)^j}{(q^2)_{j-1} (zq^{j+1})_\infty}$$

$$= \mathrm{I} + \mathrm{II}.$$

Combinatorial interpretation of II :

Let the exponent of q in the summand of II denote the number of 1's in a partition

$$\frac{z^{-j}\, q^{\overbrace{1+1+\cdots+1}^{j}}}{(1-q^2)(1-q^3)\cdots(1-q^j)} \qquad (j>0)$$

Note that the partition π generated is the one which has j ones, i.e., $\omega(\pi) = j$.

We don't pick any 1's from the denominator.

The power of z in $\frac{1}{(zq^{j+1})_\infty}$

is keeping track of the number of parts greater than j, i.e. greater than the number of 1's?

This is nothing but $\mu(\pi)$

Hence putting all of this together, we see that the power of z in II is $\mu(\pi) - \omega(\pi)$, which is the crank for $\omega(\pi) > 0$.

<u>Combinatorial interpretation of</u> I $= \frac{1-q}{(zq)_\infty}$

Note that in $\frac{1}{(zq)_\infty}$, z keeps track of the number of parts.

By conjugation, $\frac{1}{(zq)_\infty}$ generates partitions π with z keeping track of the largest part.

On the other hand, $\frac{q}{(zq)_\infty}$ generates partitions with at least one 1 & where z keeps track of the largest part, only when $n > 1$. (Explanation given later).

Hence, $\frac{1-q}{(zq)_\infty}$ generates partitions with no 1's and where power of z is the largest part, which is the defn. of crank in this case (i.e., $\omega(\pi)=0$).

Why does it fail when $n=1$?

Note that when $n=1$, that is, the number being partitioned is 1, then q^1 represents the partition. But then the power of z from the denominator should be zero, for, otherwise, you would end up getting a partition of 1, whose summands add up to a number >1, which is absurd.

But then 0, that is, the power of z is no longer the largest part of the partition.

<u>Exception is</u> <u>$n=1$ case</u>.

$$M(0,1) = N_V(0,1) = (-1)^1 = -1.$$
$$(\vec{\pi} = (1,0,0))$$

$$M(1,1) = N_V(1,1) = (-1)^0 = 1$$
$$\vec{\pi} = (0,1,0)$$

$$M(-1,1) = N_V(-1,1) = (-1)^0 = 1,$$
$$\vec{\pi} = (0,0,1)$$