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- $N(m,n)$: the number of partitions of n with rank m.
- $N(-m,n) = N(m,n)$,

since conjugation of a partition negates the rank

Thm. 42 (Dyson)
For $m>0$,

$$\sum_{n=0}^{\infty} N(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n(3n-1)}{2} + mn} (1-q^n).$$

Notation: $${}_2\phi_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n.$$

In general,

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots b_r \end{matrix} ; z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(b_1)_n (b_2)_n \cdots (b_r)_n (q)_n} z^n,$$

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n$$

where $(x)_n = x(x+1) \cdots (x+n-1)$

${}_2\phi_1$

Watson's q-analogue of Whipple's theorem

$$_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & aq^{N+1} \end{matrix} ; \frac{a^2q^{N+2}}{bcde}\right]$$

$$= \frac{(aq)_N\left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N\left(\frac{aq}{e}\right)_N}\ {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, & d, & e, & q^{-N} \\ \frac{deq^{-N}}{a}, & \frac{aq}{b}, & \frac{aq}{c} \end{matrix} ; q\right]$$

Let $b = z$, $c = z^{-1}$ & let $|q|<1$, $|q|<|z|<|q|^{-1}$.

LHS =

$$1+\sum_{n=1}^{\infty}\frac{(a)_n\{(q\sqrt{a})_n(-q\sqrt{a})_n\}(z,z^{-1},d,e)_n\,(q^{-N})_n}{\{(\sqrt{a})_n(-\sqrt{a})_n\}\left(\frac{aq}{z},aqz,\frac{aq}{d},\frac{aq}{e},aq^{N+1}\right)_n(q)_n}\left(\frac{a^2q^{N+2}}{de}\right)^n$$

- $$(wq^{-N})_n = \left(1-\frac{w}{q^N}\right)\left(1-\frac{w}{q^{N-1}}\right)\cdots\left(1-\frac{w}{q^{N-(n-1)}}\right)$$

$$= \frac{(-w)^n}{q^{Nn-\frac{n(n-1)}{2}}}\cdot\left(1-\frac{q^{N-(n-1)}}{w}\right)\left(1-\frac{q^{N-(n-2)}}{w}\right)\cdots\cdots\left(1-\frac{q^N}{w}\right)$$

$$= (-w)^n q^{\frac{n(n-1)}{2}-Nn}\ \frac{\left(\frac{q}{w}\right)_N}{\left(\frac{q}{w}\right)_{N-n}}.$$

- $$\frac{(a)_n (q\sqrt{a})_n (-q\sqrt{a})_n}{(\sqrt{a})_n (-\sqrt{a})_n}$$

$$= \frac{(a)_n (aq^2;q^2)_n}{(a;q^2)_n} = (aq)_{n-1}(1-aq^{2n}).$$

Therefore,

$$8\phi_7 = 1 + \sum_{n=1}^{\infty} \frac{(aq)_{n-1}(1-aq^{2n})(z,z^{-1},d,e)_n (-1)^n q^{\frac{n(n-1)}{2} - Nn}}{\left(\frac{aq}{z}\right)_n (aqz)_n \left(\frac{aq}{d}\right)_n \left(\frac{aq}{e}\right)_n (aq^{N+1})_n (q)_n}$$

$$\times \frac{(q)_N}{(q)_{N-n}} \cdot \left(\frac{a^2 q^{N+2}}{de}\right)^n .$$

Let $a=1$ & then let $d, e, N \to \infty$.

- $$\frac{(d)_n}{d^n} = \frac{(1-d)}{d}\frac{(1-dq)}{d} \cdots \frac{(1-dq^{n-1})}{d}$$

$$= \left(\frac{1}{d}-1\right)\left(\frac{1}{d}-q\right)\cdots\left(\frac{1}{d}-q^{n-1}\right)$$

$$\longrightarrow (-1)(-q)\cdots(-q^{n-1})$$

$$= (-1)^n q^{n(n-1)/2} \quad \text{as } d \to \infty$$

$\Rightarrow$

$$8\phi_7 = 1 + \sum_{n=1}^{\infty} \frac{(1+q^n)(z)_n (z^{-1})_n \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^2}{(q/z)_n (qz)_n}$$

$$\times (-1)^n q^{\frac{n(n-1)}{2}} \cdot q^{2n}$$

$$\frac{(z)_n(z^{-1})_n}{(zq)_n(z^{-1}q)_n} = \frac{(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}$$

$\Rightarrow$ LHS

$$= 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(1+q^n)(-1)^n q^{\frac{3n(n-1)}{2}+2n}}{(1-zq^n)(1-z^{-1}q^n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(1+q^n)(-1)^n q^{\frac{n(3n+1)}{2}}}{(1-zq^n)(1-z^{-1}q^n)}.$$

Note that

$$\frac{q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = 1 - \frac{(1-q^n)}{(1+q^n)}\left[\frac{1}{1-zq^n} + \frac{1}{1-z^{-1}q^n} - 1\right]$$

(since RHS $= 1 - \frac{(1-q^n)}{1+q^n}\left[\frac{1}{1-zq^n} + \frac{z^{-1}q^n}{1-z^{-1}q^n}\right]$

$$= 1 - \frac{(1-q^n)}{1+q^n}\left[\frac{1 - \cancel{z^{-1}q^n} + \cancel{z^{-1}q^n} - q^{2n}}{(1-zq^n)(1-z^{-1}q^n)}\right]$$

$$= 1 - \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^n)}$$

$$= \frac{\cancel{1} - z^{-1}q^n - zq^n + \cancel{q^{2n}} - \cancel{1} + 2q^n - \cancel{q^{2n}}}{(1-zq^n)(1-z^{-1}q^n)}$$

$$= \frac{q^n(2 - z - z^{-1})}{(1 - zq^n)(1 - z^{-1}q^n)} = \frac{q^n(1-z)(1-z^{-1})}{(1 - zq^n)(1 - z^{-1}q^n)}$$

$\Rightarrow$ LHS =

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n)\left[1 - \frac{(1-q^n)}{1+q^n}\left\{\frac{1}{1-zq^n} + \frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}\right]$$

1st expression on

$$\text{RHS} = \frac{(aq)_N \left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N \left(\frac{aq}{e}\right)_N} \longrightarrow (q)_\infty \text{ as } d, e, N \to \infty \text{ and } a = 1$$

2nd expression on RHS

$$= {}_4\varphi_3 \left[\begin{matrix} \frac{aq}{bc}, d, e, q^{-N} \\ \frac{deq^{-N}}{a}, \frac{aq}{b}, \frac{aq}{c} \end{matrix} ; q\right]$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(q)_n} \left(\frac{q}{de}\right)_{N-n}}{\cancel{(-1)^n} \cancel{q^{\frac{n(n-1)}{2}}} {}^{-nN} \left(\frac{q}{de}\right)_N} \left(\frac{(d)_n}{d^n}\right)\left(\frac{(e)_n}{e^n}\right)$$

$$\times \frac{\cancel{(-1)^n} \cancel{q^{\frac{n(n-1)}{2}}} {}^{-nN} (q)_N}{\left(\frac{q}{z}\right)_n (zq)_n (q)_{N-n}} \frac{q^n}{\cancel{(q)_n}}$$

$$\xrightarrow[\to\infty]{d,e,N} \sum_{n=0}^{\infty} \frac{\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^2 q^n}{(zq)_n (z^{-1}q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

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$$\Rightarrow \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

$$= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n) \left[1 - \frac{(1-q^n)}{1+q^n} \left\{ \frac{1}{1-zq^n} + \frac{z^{-1}q^n}{1-z^{-1}q^n} \right\} \right] \right\}$$

$$= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n) \right.$$

$$\left. - \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \frac{(1-q^n)}{1-zq^n} - \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \frac{(1-q^n) z^{-1} q^n}{1-z^{-1}q^n} \right\}$$

Note that

$$1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} (1+q^n)$$